Logarithmic Limit

Recall: $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ $\quad x \oplus y = \max(x,y)$ & $x \otimes y = x + y$

$\mathbb{T}$ is a semi-field.

Consider $(\mathbb{R} \cup \{-\infty\}, \oplus_t, \otimes)$ with $t > 1$ $\quad x \oplus_t y := \log_t(t^x + t^y)$, $\quad x \otimes y := x + y$

Semifield

Iso to the semifield $(\mathbb{R}_{\geq 0}, +, \cdot)$

$\rightsquigarrow$ Take $\log_t$ of $\max(t^x, t^y) \leq t^x + t^y \leq 2\max(t^x, t^y)$ to get

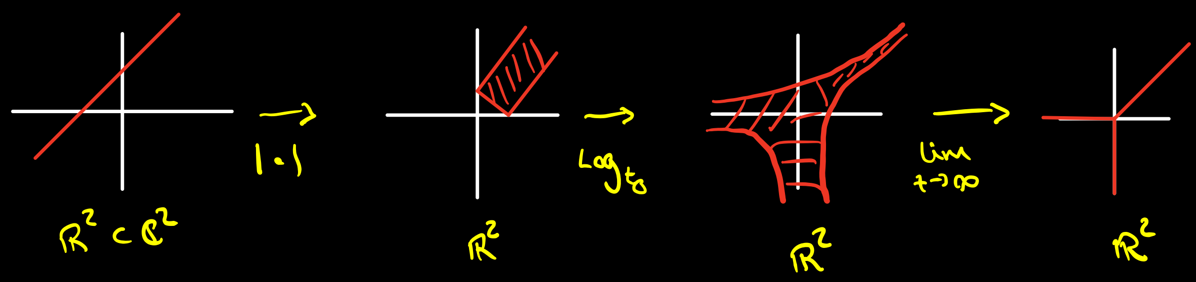
$$\max(x,y) \leq x \oplus_t y \leq \max(x,y) + \underbrace{\log_t(2)}_{= \frac{\ln(2)}{\ln(t)}}$$

$$\lim_{t \to \infty} x \oplus_t y = x \oplus y \quad \text{(tropical sum)}$$

$\rightsquigarrow$ Tropical varieties can arise as limits via $\log_t$.

Example:

$x - y + 1 = 0$ in $\mathbb{C}^2$.



$\rightsquigarrow$ Given a variety $X \subset \mathbb{C}^2$, get a tropical variety in $\mathbb{R}^2$.

Remark

The images above under the map Log_t for different t's are called **Amoebas**. One can study the tropical variety by studying these.

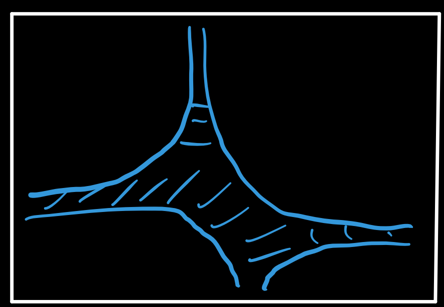
Example $f = (1 + x + y) + t\,xy$ family of quadrics $X_t = \{f_t = 0\} \subset \mathbb{C}^2$

$$A_t = \mathrm{Log}_t(X_t) \subset \mathbb{R}^2$$

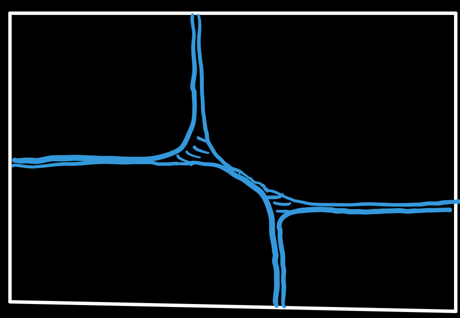
$\uparrow$

Amoeba base t.

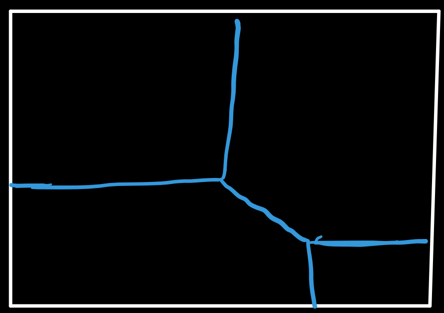
t small



t large



$t \to \infty$



3 roads to a tropical variety

$K = \mathbb{C}\{\{t\}\} = \bigcup_{n\geq 1} \mathbb{C}((t^{1/n}))$ The field of Puiseux series.

E.g.: $3t^{-2/3} + 4 + t + t^4 + \cdots$

Valuation: $\mathrm{val}: K^* \longrightarrow \mathbb{R}$

$$\sum_{i=k}^{\infty} a_i t^{i/n} \mapsto k/n$$

- $\mathrm{Val}(xy) = \mathrm{Val}(x) + \mathrm{Val}(y)$
- $\mathrm{Val}(x+y) \geq \min\{\mathrm{Val}(x), \mathrm{val}(y)\}$

(Equality when $\mathrm{val}(x) \neq \mathrm{Val}(y)$)

Remark: K is algebraically closed. The algebraic closure of $\mathbb{C}((t))$ is K.

Spec $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

$\in K = \mathbb{C}\{\{t\}\}$

Let $X \subset (K^*)^n$ be a hypersurface. i.e. the zero set $\{f=0\}$ for some $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$ (Multi-index notation)

$(u_1, \dots, u_n)$ $\quad x_1^{u_1} \cdots x_n^{u_n}$

- For any point $(y_1, \dots y_n) \in (K^*)^n$ we get $(\mathrm{Val}(y_1), \dots, \mathrm{Val}(y_n)) \in \mathbb{R}^n$

Def(1) The tropicalization of X is the closure in $\mathbb{R}^n$ of the set

$$X^{trop} = \overline{\{(\mathrm{Val}(y_1), \dots, \mathrm{Val}(y_n)) : f(y_1, \dots y_n) = 0\}}$$

Remark: $|\cdot| := e^{-\mathrm{Val}(\cdot)}$ defines an absolute value on K.

Then

$$X^{trop} = \{-(\log(\underset{\mathrm{Val}(y_1)}{\overset{\shortparallel}{|y_1|}}), \dots, \log(|y_n|)) \mid f(y_1, \dots y_n)\}$$

Note: Very different from the usual (archimedean) absolute value.

- From f we can also construct a tropical polynomial:

$$\mathrm{trop}(f) : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$w \longmapsto \min_{u \in \mathbb{Z}^n} \left(\mathrm{val}(c_u) + u \cdot w \right)$$

$$\mathrm{trop}(f) = \bigoplus_u \mathrm{val}(c_u) \otimes x_1^{u_1} \otimes \dots \otimes x_n^{u_n}$$ as a tropical poly.

Def 2: The tropicalization of X is the locus

$$V(\mathrm{trop}(f)) = \{ w \in \mathbb{R}^n \mid \text{minimum of } \mathrm{trop}(f)(w) \text{ achieved atleast twice} \}$$

Def: For $w \in \mathbb{R}^n$ define

Picks out the coeff in front of $t^{\mathrm{val}(c_u)}$

$$\mathrm{in}_w(f) = \sum_{\substack{u \in \mathbb{Z}^n \text{ s.t} \\ \mathrm{val}(c_u) + w \cdot u = \mathrm{trop}(f)(w)}} \overline{t^{-\mathrm{val}(c_u)} c_u} \; x^u$$

$\mathrm{In}_w(f)$ is called an initial term.

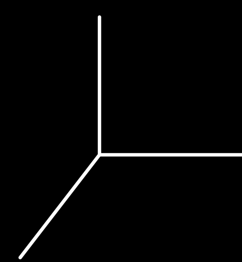
Remark: $\mathrm{In}_w(f)$ is the sum of the residue of the terms in f that achives its minimum.

Def 3: The **tropicalization** of X is the set of $w \in \mathbb{R}^n$ with $(\mathrm{in}_w(f)) \neq (1)$ in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Example:

$f = x - y + 1 \in K[x^{\pm 1}, y^{\pm 1}]$

- $\mathrm{trop}(f) = 0 \oplus x \oplus y$ so $V(\mathrm{trop}(f))$ is
$= \min(0, x, y)$



- The zeros of f are $(z, z+1)$ with $z \in K \setminus \{0, -1\}$.

Recall $\mathrm{Val}(z+1) = \min(\mathrm{Val}(z), \mathrm{Val}(1)) = \min(\mathrm{Val}(z), 0)$

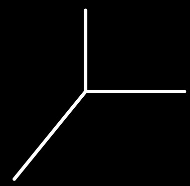
When $\mathrm{Val}(z) \neq 0$.

$\mathrm{Val}(z) = 0 \Rightarrow \mathrm{Val}(z+1) \geq 0$.

In total:

$$(\mathrm{Val}(z), \mathrm{Val}(z+1)) = \begin{cases} (\mathrm{Val}(z), 0) & \mathrm{Val}(z) > 0 \\ (\mathrm{Val}(z), \mathrm{Val}(z)) & \mathrm{Val}(z) < 0 \\ (0, \mathrm{Val}(z+1)) & \mathrm{Val}(z) = 0 \\ (0, 0) & \text{o/w} \end{cases}$$

This is the line



- For $w=(a,b)$ we have $\mathrm{trop}(f)(w) = \min(a, b, 0)$

So $\mathrm{in}_w(f)$ not monomial $\iff$ $a=0$, $b=0$, or $a=b<0$

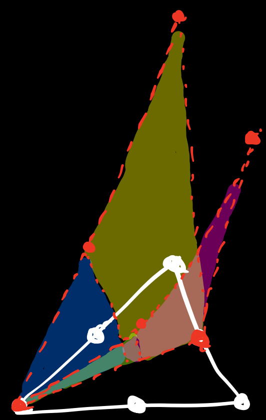
which is the line.

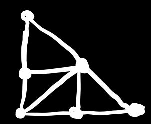
$\mathrm{in}_{(a,0)}(f) = -y+1$ $\quad$ $\mathrm{in}_{(0,b)}(f) = x+1$ $\quad$ $\mathrm{in}_{(0,0)}(f) = x-y+1$

$\underset{a,b<0}{\mathrm{in}_{(a,b)}(f)} = x-y.$

Example

$f = (3t^3+5t^2)xy^{-1} + 8t^2y^{-1} + 4t^{-2}$. Draw $\mathrm{trop}(V(f))$.





Example:

$f = 1 + x + y + txy.$

Theorem (Kapranov)

For a Laurent polynomial $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ the following sets are equal:

1) $V(\operatorname{trop}(f))$

2) $\{ w \in \mathbb{R}^n \mid \operatorname{in}_w(f) \text{ not monomial} \}$

3) $\overline{\{ (\operatorname{val}(y_1), \dots, \operatorname{val}(y_n)) \mid f(y_1, \dots, y_n) = 0 \}}$

Remarks

- Can replace K with any (non-trivial) valued field.

- For every $w \in \mathrm{Trop}(V(f)) \rightsquigarrow$ initial form $\mathrm{In}_w(f)$
  $\rightsquigarrow$ Degeneration of $V(f)$ to $V(\mathrm{In}_w(f))$

Let $m = \min\{u \cdot w \mid c_u \neq 0\}$

$t^{-m} f(t^{w_1} x_1, \dots, t^{w_n} x_n) \in K[t][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ defines a

variety $\mathcal{X} \xrightarrow{g} \mathbb{A}^1$ where $g^{-1}(0) = V(\mathrm{In}_w(f))$

and general fiber isomorphic to $f(x_1, \dots, x_n) = 0$.

Upshot: $\mathrm{Trop}(V(f))$ parametrizes degenerations of $V(f)$.

The general case:

Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an ideal & $X = V(I)$

- $\mathrm{trop}(X) := \bigcap_{f \in I} \mathrm{trop}(V(f))$
- $\mathrm{In}_w(I) := \{\mathrm{In}_w(f) \mid f \in I\}$

Warning: $I = (f_1, \dots, f_n) \not\Rightarrow \operatorname{trop}(I) = (\operatorname{trop}(f_1), \dots, \operatorname{trop}(f_n))$

Theorem (Fundamental thm of tropical geometry)

Notation as above. The following sets coincide.

1) $\operatorname{trop}(X)$

2) $\{ w \in \mathbb{R}^n \mid \operatorname{In}_w(I) \neq (1) \}$

3) $\overline{\{ (\operatorname{val}(y_1), \dots, \operatorname{val}(y_n)) \mid (y_1, \dots, y_n) \in X \}}$

Weights?

Consider a curve $X = V(f)$ $\quad f = \sum c_u x^u$.

In all three cases we obtain weights:

1) As we did last time (lattice points in dual polygon)

2) # Components (w/ multiplicity) in $V(\operatorname{In}_w(f))$

3) For $\dim(X) = 0$, look at size of fibers $X \xrightarrow{\operatorname{val}} \operatorname{trop}(X)$

Generalize.

Exercises

1) Draw $\operatorname{trop}(V(f))$ for

- $f = t^{3}x + (t + 3t^{2} + 5t^{4})y + t^{-2}$
- $f = t^{3}x^{2} + xy + ty^{2} + tx + y + 1$

2) Show that $\operatorname{Val} : K^{*} \to \mathbb{Q}$ satisfies

$$\operatorname{Val}(x+y) = \min(\operatorname{Val}(x), \operatorname{Val}(y)) \quad \text{for } \operatorname{Val}(x) \neq \operatorname{Val}(y)$$

3) Define $\operatorname{Val} : K \to \mathbb{Q} \cup \{\infty\}$ by $\operatorname{Val}(0) := \infty$.

Set $|\cdot| = e^{-\operatorname{Val}(\cdot)}$. This satisfies

1) $|x| = 0 \iff x = 0$
2) $|xy| = |x||y|$
3) $|x+y| \leq \max(|x|, |y|)$

Let $B(r,x) = \{ y \in K \mid |x-y| \leq r \}$ be the ball of radius r centered around x.

Show: Any point $y \in B(r,x)$ is the center of the ball.

i.e. $B(r,x) = B(r,y)$ whenever $|x-y| \leq r$.

4) Prove that a tropical hypersurface $V \subset \mathbb{R}^n$ with Newton polytope the standard n-simplex of size d has at most d^n vertices, and that V is non-singular iff equality holds.

5) Let $S \subset \mathbb{R}^3$ be a non-singular tropical surface with Newton polytope $2\Delta_3$. Show that S has a unique compact facet.

(Hint: One must prove that the dual subdivision has a unique edge not contained in the boundary; use that the tetrahedron has euler characteristic 1)

6) Construct a polynomial f such that $\operatorname{trop}(V(f))$ is a fan, and f has a coefficient with non-zero valuation

7) What is the largest multiplicity of any edge in the tropicalization of a plane curve of degree d.

How about surfaces in 3-space?