Matroids and their Bergman fan

- First introduced in Hassler Whitney's paper:
  On the abstract properties of linear dependence (1935)
- combinatorial objects abstracting the notion of independence
- Ex:
  - linear algebra
    linearly independent vectors
  - Graph theory
    the forests of some complete graph
- Can come from vector configurations or du hyperplane arrangements (or)
- Exists several equivalent def (7, 8 ?)

Outline:

- Introduce hyperplane arrangements and a rank function
- Define Matroids and:
  - Flats, lattice of flats, flags
- Define Bergman fan

  $\downarrow$

  How it is related to tropical geometry!!
- Examples

Hyperplane arrangement

→ not ⊂ in any coordinate hyperplane

Given a linear subspace $\mathcal{L} \subseteq K^{n+1}$ of dim $d+1$

$$\begin{array}{ccc} \mathcal{L} & \subseteq & K^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{L}) & \subseteq & \mathbb{P}^n \end{array}$$

can define a hyperplane arrangement $\mathcal{A}$

$\dim(H_i) = \dim(\mathbb{P}(\mathcal{L})) - 1$

$$\mathcal{A} = \{H_0, \dots, H_n\} \quad H_i = \mathbb{P}(\mathcal{L}) \cap \underbrace{(x_i = 0)}_{\cap\ \mathbb{P}^n}$$

ex: lines in the plane, planes in 3-dim spaces.

Can define a rank function r on $\mathcal{A} \subseteq \mathbb{P}(\mathcal{L})$

- let E $\{0,1,\dots,n\}$ be the ground set
- For $S \subseteq E$, let

$$r(S) := \operatorname{codim}\Big(\Big(\bigcap_{i \in S} H_i \subseteq \mathbb{P}(\mathcal{L})\Big)\Big)$$

$$\dim(\mathbb{P}(\mathcal{L})) - \dim\Big(\bigcap_{i \in S} H_i\Big)$$

$$r(\{0,2,5\}) = \operatorname{codim}(H_0 \cap H_2 \cap H_5) = 2$$

$$r(\{0,5,4\}) = \operatorname{codim}(H_0 \cap H_5 \cap H_4 = \emptyset) = 3$$

$$r(E) = \operatorname{codim}\Big(\bigcap_{i \in E} H_i\Big) = 3$$

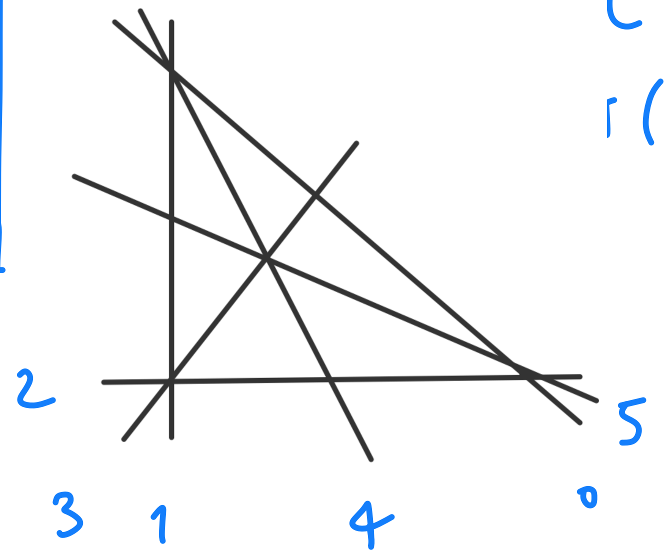
Ex rank function on a hyperplane-arrangement

let $\mathcal{L} \subseteq K^6$ of dim 3

$\mathbb{P}(\mathcal{L}) \subseteq \mathbb{P}^5$ of dim 2

$$\mathcal{A} := \{H_0, \dots, H_5\} \subseteq \mathbb{P}(\mathcal{L}) \quad H_i = \mathbb{P}(\mathcal{L}) \cap \overset{\mathbb{P}^5 \cup}{X_i = 0}$$

$E = \{0,1,\dots,5\}$



$$r(\{0\}) = \operatorname{codim}(H_0) = \dim(\mathbb{P}(\mathcal{L})) - \dim(H_0) = 2 - 1 = 1$$

$$r(\{i\}) = 1$$

$$r(\{0,2\}) = \operatorname{codim}(H_0 \cap H_2) = 2$$

$$r(\{i,j\}) = 2$$

Def: A matroid on E of rank $d+1$ is a function

$$r : 2^E \to \mathbb{Z}$$

Satisfying for $S \subseteq E$ ✓

(1) $0 \leq r(S) \leq |S|$ ✓

(2) $S \subseteq U$ implies $r(S) \leq r(U)$ ✓

(3) $r(S \cup U) + r(S \cap U) \leq r(S) + r(U)$ and ✓

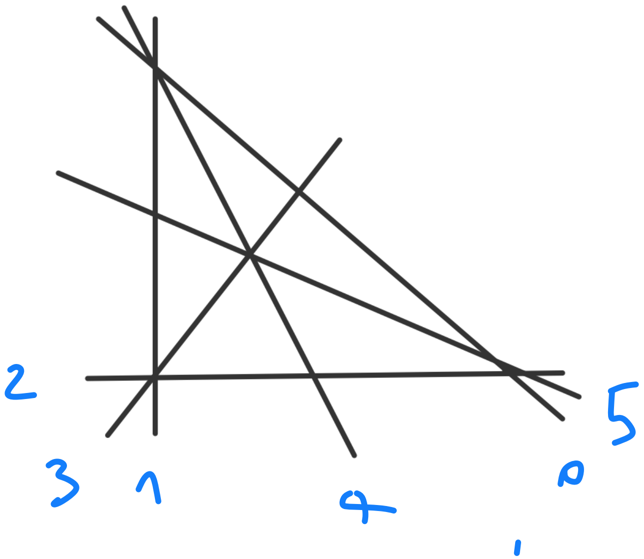
(4) $r(\{0, \dots, n\}) = d+1$ ✓

Our rank function is a matroid

- Independent subsets $I \subset E$ s.t. $r(I) = I$
  - ↳ Define matroids in term of independent sets.

Def: A *flat* of r is a subset $S \subseteq E$ such that for any $j \in E$, $j \notin S$, $r(S \cup \{j\}) > r(S)$

Ex



$\mathcal{A} \subseteq \mathbb{P}(\mathbb{L})$ of rank 2

- $\emptyset$ is a flat $r(\emptyset) = \operatorname{codim}(\mathbb{P}(\mathbb{L})) = 0$

  $r(\emptyset \cup \{i\}) = 1$

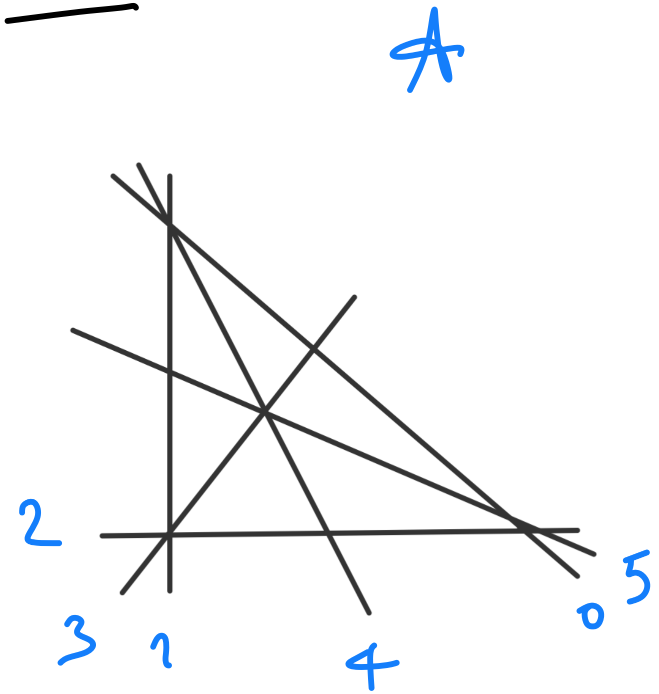
- $\{i\}$ is a flat $r(\{i\}) = 1$

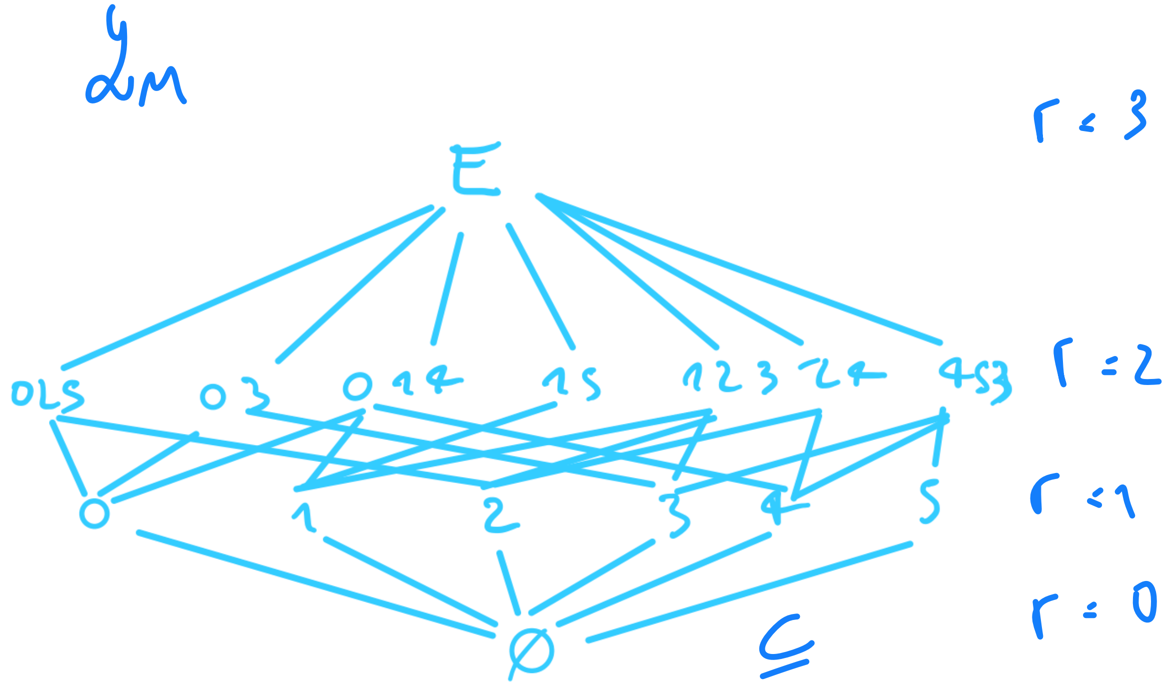
  $r(\{i\} \cup \{j\}) = r(\{i,j\}) = 2$
  $i \neq j$ $\quad$ $i \neq j$

- $r(\{0 5\}) = 2 = r(\{0 5 2\})$
- $\{4 2\}$ is a flat
- E is a flat

Flats form a partially ordered, the lattice of flats $\mathcal{L}_M$ of a matroid of M that we can represent graphically

Ex



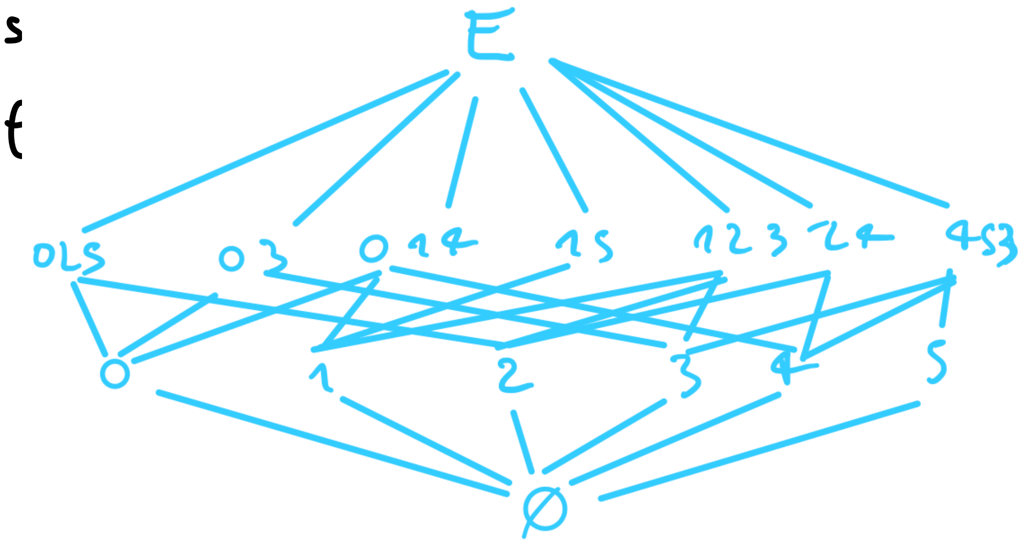


<u>Def</u>: A matroid is a collection of subsets $\mathcal{F}$ of a set E satisfying the following conditions

(1) $E \in \mathcal{F}$

(2) If $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$, and

(3) If $F \in \mathcal{F}$ and $\{F_1, F_2, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ properly containing F then the sets $F_1 \setminus F, F_2 \setminus F, \ldots, F_k \setminus F$ form a partition of $E \setminus F$



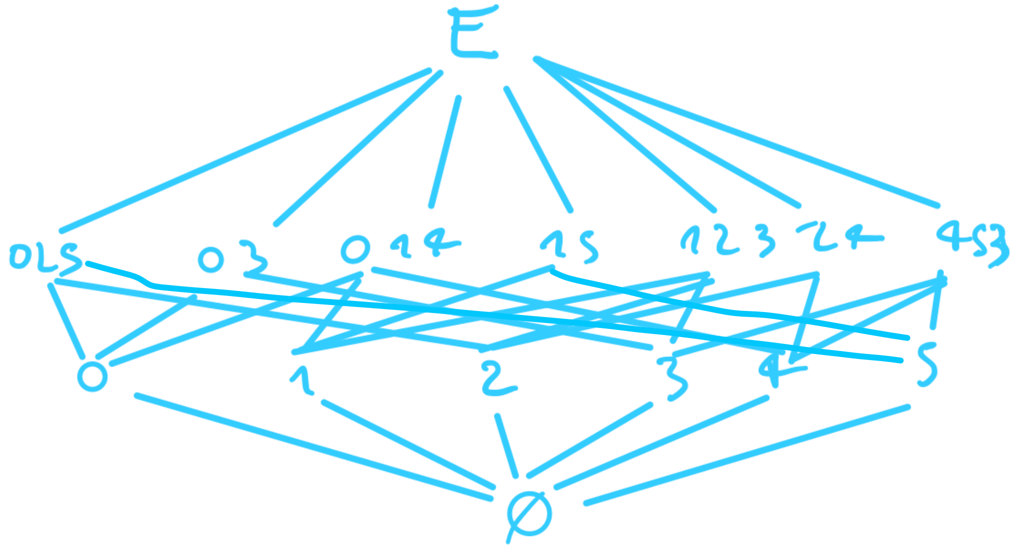
Def: A k-step flag of proper flats is a sequence of proper flats ordered by inclusion

$$F_\bullet = \{ \emptyset \subsetneq F_1 \subseteq \cdots \subseteq F_k \subsetneq E \}$$

Ex:

$F_0 = \{ \emptyset, E \}$

$F_1 = \{ \emptyset, \{0\}, \{0\}\}, E \}$



The lattice of flats of a matroid M can be represented by a polyhedral fan called its Bergman fan

Preliminaries/Notations:

- Given a set of vectors $\{v_0, \dots, v_n\}$ denote $\text{cone}(v_0, \dots, v_n) := \{\sum_{i=0}^{n} \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$ the cone that they generate

- $\{e_i \mid i \in E\}$ be the standard basis on $\mathbb{Z}^E$
- $e_F = \sum_{i \in F} e_i$ for a flat F → ex -
- $N := \mathbb{Z}^E / \sum_{i \in E} e_i$
- u_i / u_F the image of e_i / e_F

u_F will be the rays in the fan!

Def: let M be a matroid of rank $d+1$ on a ground set E. with the notation as above, the Bergman fan Σ_M is the pure d-dimensional polyhedral fan in $N_\mathbb{R} = N \otimes \mathbb{R}$ that is comprised of cones

$$\sigma_{\mathcal{F}} := \mathrm{cone}(u_{F_1}, u_{F_2}, \dots, u_{F_k}) \subset N_\mathbb{R}$$

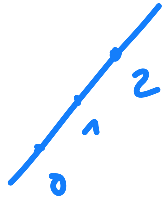
for each chain of flats

$$\mathcal{F} : \emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subseteq E$$

in $\mathcal{L}_M$

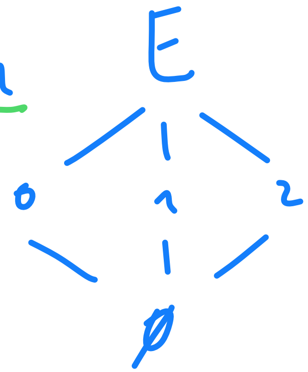
Ex: let $\mathcal{L}$ be a line not contained in any of the coordinate-axis of $\mathbb{P}^2$

$\mathcal{A} = \{H_0, H_1, H_2\} \subseteq \mathcal{L}$



$r(\emptyset) = 0$
$r(i) = 1$
$r(ij) = 2$
$r(E) = 2$

$\mathcal{L}_M$



Flags

$\mathcal{F}_a = \{\emptyset, E\}$

$\mathcal{F}_b = \{\emptyset, \{0\}, E\}$

$\mathcal{F}_c = \{\emptyset, \{1\}, E\}$

$\mathcal{F}_d = \{\emptyset, \{2\}, E\}$

Expecting a 1-dim polyhedral fan in $N_\mathbb{R} = \mathbb{Z}^3/(e_0+e_1+e_2) \otimes \mathbb{R}$

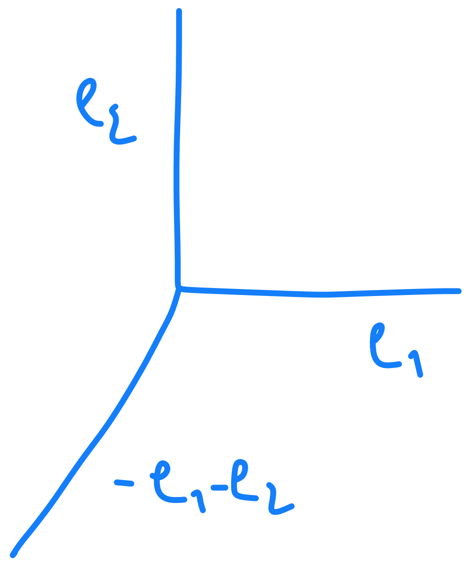
$\sigma_{\mathcal{F}_a}$: $\mathrm{cone}(u_\emptyset, u_E)$
$u_E = e_0 + e_1 + e_2 = 0$ in N

$\sigma_{\mathcal{F}_b}$: $\mathrm{cone}(u_\emptyset, u_0, u_E)$
$u_0 = e_0 / e_0 + e_1 + e_2 = -e_1 - e_2$

$\sigma_{\mathcal{F}_c} = \mathrm{cone}(u_\emptyset, u_1, u_E)$
$= \mathrm{cone}(e_1)$

$\sigma_{\mathcal{F}_d} = \mathrm{cone}(u_\emptyset, u_2, u_E)$
$= \mathrm{cone}(e_2$

The Bergman fan Σ_M is comprised of the cones:
cone(e_1), cone(e_2),
cone$(-e_1-e_2)$ in $N \otimes \mathbb{R}$



the tropical line!

Theorem (Sturmfels 2002)

The tropicalization of a linear space $V \subseteq \mathbb{C}^n$ is the Bergman fan $\Sigma_{M(V)}$

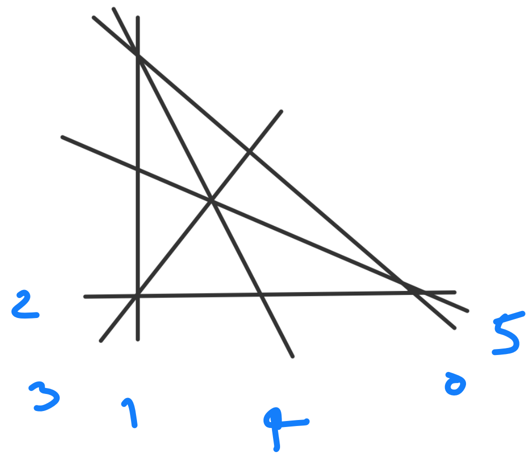
$$\lim_{t \to \infty} \log_t(V)$$

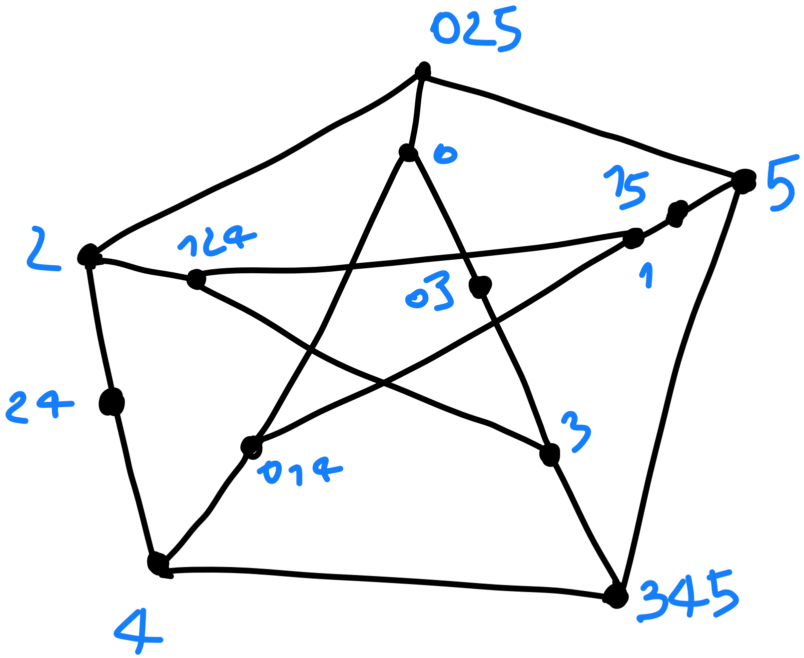
- Different linear spaces have the same matroid

- Remembers some of the invariants

Ex) The Bergman fan of $\mathbb{P}(\mathcal{L}) \subseteq \mathbb{P}^5$ is a 2-dim fan in a 5-dim space.

↳ Can intersect with a sphere containing the vertex of the fan





13 Rays ⟶ points (flats)

18 2-dim cones ⟶ lines (flags)

Bergman complex

Proposition (Sturmfels 2002):
The Bergman fan of a matroid is a balanced fan when equipped with weights equal to 1 on all its top-dimensional cones