UiO : Department of Mathematics
University of Oslo

Torelli theorems

Trial Lecture

Edvard Aksnes

March 7, 2024

Table of contents

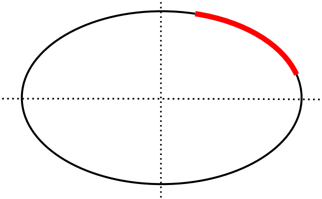
1 **Background and prerequisites**

2 **Curves**

3 **K3 surfaces and other generalizations**

Background and prerequisites

Abelian integrals



$$f(x,y) = x^2/a^2 + y^2/b^2 - 1 = 0$$

$$\int \sqrt{a^2\cos(\theta)^2 + b^2\sin(\theta)^2}d\theta$$
$$= \int \frac{(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$$

From a rational function $R(x,y)$ and polynomial identity $f(x,y) = 0$ for f of degree $d > 2$, we have an abelian integral:

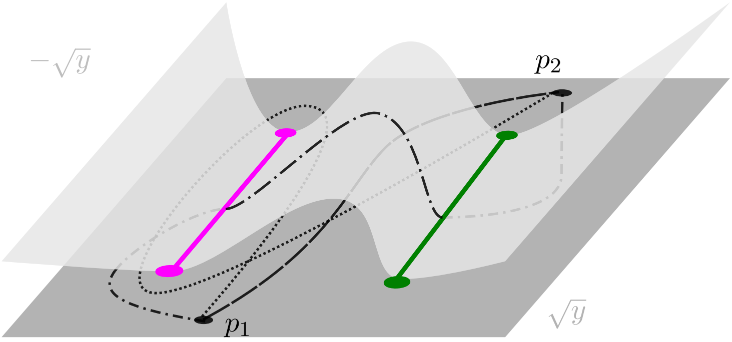
$$\int R(x,y)dx.$$

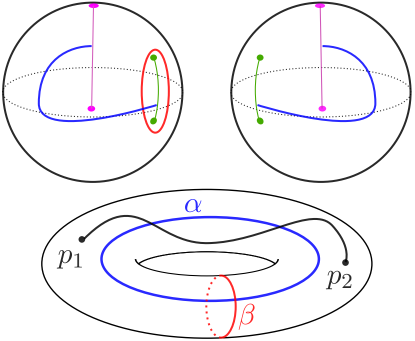
More generally, what about when the variables are complex?

Example: Letting $f(x,y) = x^3 + ax^2 + bx + c - y^2 = 0$, $R(x,y) = 1/y$:

$$\int_{p_1}^{p_2} \frac{dx}{y} = \int_{p_1}^{p_2} \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}$$

Must choose branches $\pm\sqrt{}$ and glue along branch cuts between the roots of $x^3 - ax^2 + bx + c$ and ∞.



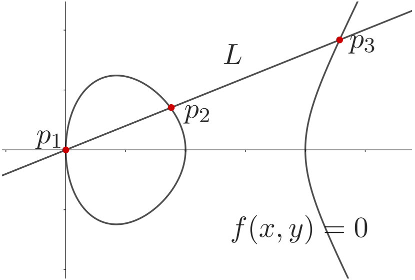


On the Riemann Surface S of $f(x, y)$, the integral

$$\int_{p_1}^{p_2} \frac{dx}{y}$$

is well defined up to periods

$$\int_{\alpha} \frac{dx}{y} \quad \text{and} \quad \int_{\beta} \frac{dx}{y}.$$

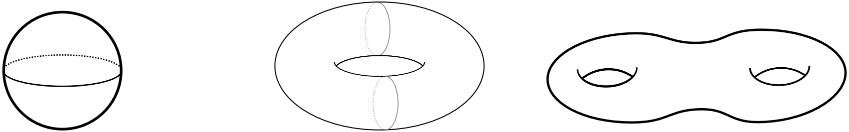


Theorem (Abel's theorem)

$$\sum_{i=1}^{3} \int_{p_0}^{p_i} \frac{dx}{y} \equiv K \mod \textit{periods}$$

Distinguishing between varieties

How can we tell the difference between two Riemann surfaces?



$\deg f \leq 2$ $\qquad$ $\deg f = 3$ $\qquad$ $\deg f > 3$

More generally, an algebraic variety X is a set

$$X = \{(x_1, \ldots, x_n) \mid f_1(x_1, \ldots, x_n) = 0,\ \ldots,\ f_k(x_1, \ldots, x_n) = 0\}.$$

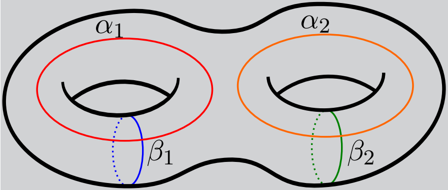
Torelli-type theorems

A Torelli-type theorem transforms the problem distinguishing varieties into a linear algebra problem.

Singular homology

The homology groups $H_r(X; \mathbb{Z})$ are topological invariants of spaces, "detecting the holes" in our space.

Example (Homology of a two-holed torus)



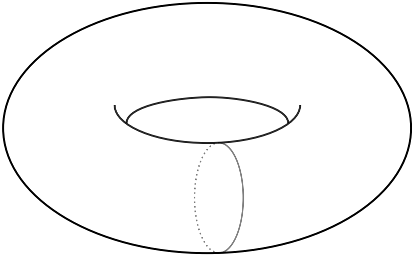
$$H_1(S; \mathbb{Z}) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \rangle_{\mathbb{Z}} \cong \mathbb{Z}^4$$

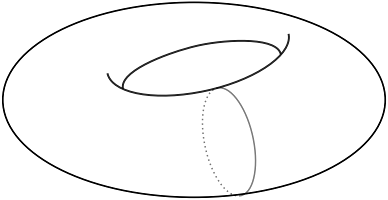
$$Q\colon H_1 \otimes H_1 \to \mathbb{Z}$$

$$Q = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

The cohomology $H^r(X; \mathbb{Z})$ is defined dually to homology, and (in our cases) the bilinear pairing Q induces $H^r(X; \mathbb{Z}) \cong \mathrm{Hom}(H_r(X; \mathbb{Z}), \mathbb{Z})$.

But homology is not sufficient





$$y^2 = x(x-1)(x-2)$$

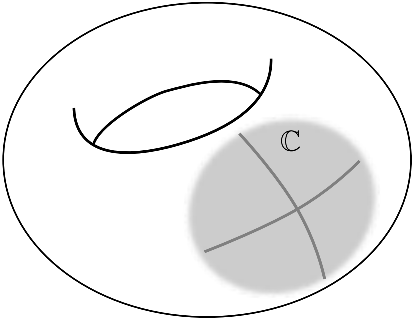
$$H_1(S;\mathbb{Z}) = \mathbb{Z}^2$$

$$y^2 = x(x-1)(x-1-i)$$

$$H_1(S';\mathbb{Z}) = \mathbb{Z}^2$$

Complex analysis

A Riemann surface S is also a complex manifold.



All the machinery of complex analysis transfers to S:

- Complex coordinates
  $x + iy = z$.
- $f : S \to \mathbb{C}$ can be holomorphic
  $\partial f / \partial \overline{z} = 0$.

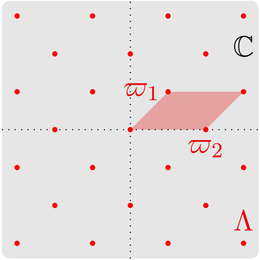
Holomorphic 1-forms $H^0(S; \Omega^1) \sim \{a(z)dz \mid \overline{\partial} a = 0\}$

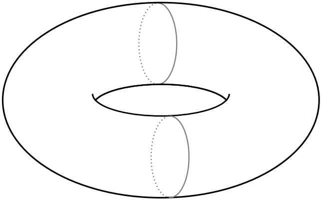
Periods determine elliptic curves

Pick bases $\langle \alpha, \beta \rangle = H^1(E;\mathbb{Z})$ and $\langle \omega \rangle = H^0(E;\Omega^1)$, and compute the periods

$$\varpi_1 = \int_\alpha \omega, \qquad \varpi_2 = \int_\beta \omega$$

are generators of a lattice $\Lambda \subset \mathbb{C}$.





$$E \cong \mathrm{Jac}(C)$$

Curves

Torelli theorem for curves

Pick compatible bases:

$$\begin{aligned} H_1(C;\mathbb{Z}) &= \langle \gamma_1, \dots, \gamma_{2g} \rangle_{\mathbb{Z}} = \Lambda, \\ H^0(C;\Omega^1) &= \langle \omega_1, \dots, \omega_g \rangle_{\mathbb{C}} = \mathbb{C}^g \end{aligned} \qquad \text{with} \quad \int_{\gamma_i} \omega_j = \delta_{ij}, \quad 0 \leq i,j \leq g.$$

One can construct the Jacobian of the curve

$$\mathrm{Jac}(C) := H^0(C;\Omega^1)^* / H_1(C;\mathbb{Z}) = \mathbb{C}^g / \Lambda.$$

Theorem (Torelli (1913), Andreotti (1958))

Two complex projective curves C and C' are isomorphic if and only if their polarized Jacobians $(\mathrm{Jac}(C), \Theta_C)$ and $(\mathrm{Jac}(C'), \Theta_{C'})$ are isomorphic.

Theta divisor

Since $\mathrm{Jac}(C)$ is a torus:

$$H^2(\mathrm{Jac}(C);\mathbb{Z}) \cong \wedge^2 H^1(\mathrm{Jac}(C);\mathbb{Z}) \cong \wedge^2 H^1(C;\mathbb{Z})$$

$Q\colon H_1(C;\mathbb{Z}) \otimes H_1(C;\mathbb{Z}) \to \mathbb{Z}$ induces a class $\omega \in H^2(\mathrm{Jac}(C);\mathbb{Z})$.

This gives a *principal polarization*, making $\mathrm{Jac}(C)$ an *abelian variety*.

Abelian variety theory:

Line bundle L on $\mathrm{Jac}(C)$ with prescribed $c_1(L)$	$\leftrightarrow$	*multipliers* on $\mathbb{C}^g$
Sections of L	$\leftrightarrow$	*theta functions* on $\mathbb{C}^g$

In particular, ω gives us a unique divisor Θ, the Theta divisor of C.

Embedding the curve

Picking a base point $p_0 \in C$, we can define the Abel-Jacobi map:

$$\mu \colon C \to \mathrm{Jac}(C)$$
$$p \mapsto \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right)$$

Which can be generalized to $\mu_{(g-1)} \colon C^{(g-1)} \to \mathrm{Jac}(C)$

$$(p_1, \dots, p_{g-1}) \mapsto \sum_{i=1}^{g-1} \mu(p_i).$$

Let $W_{(g-1)} = \mu_{(g-1)}(C^{(g-1)})$ be the image.

Theorem (Riemann's theorem)

$\Theta = W_{(g-1)} + \dots$ *as divisors on* $\mathrm{Jac}(C)$.

points of Θ $\leftrightarrow$ lists of points $p_1, \dots, p_{g-1} \in C$

Gauss Map

$$\mathrm{Jac}(C) = H^0(C, \Omega^1)^*/H_1(C;\mathbb{Z}) \quad \rightsquigarrow \quad T_x(\mathrm{Jac}(C)) \cong H^0(C,\Omega^1)^*$$

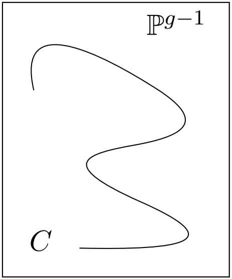
For any subvariety $X \subseteq \mathrm{Jac}(C)$ of dimension d, let X^{sm} define:

$$\begin{aligned} \mathcal{G}_X \colon X^{\mathrm{sm}} &\to \mathrm{Grass}(d, g-1) \\ x &\mapsto T_x X \subseteq T_x \mathrm{Jac}(C) \cong \mathbb{C}^g \end{aligned}$$

By $\mu\colon C \to \mathrm{Jac}(C)$, view $C \subseteq \mathrm{Jac}(C)$ as a subvariety.

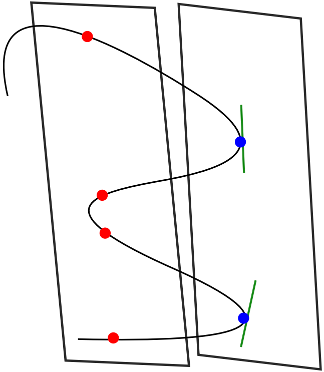
$$\begin{aligned} \mathcal{G}_C \colon C &\to \mathrm{Grass}(1, g-1) \cong \mathbb{P}^{g-1} \\ p &\mapsto (\omega_1(p), \dots, \omega_g(p)) \end{aligned}$$

is the canonical embedding.



Branching locus of the Gauss map

$$\mathcal{G}_\Theta \colon \Theta^{\mathrm{sm}} \to \mathrm{Grass}(g-2, g-1) \cong (\mathbb{P}^{g-1})^*$$



Riemann singularity theorem:

$x \in \Theta$ smooth

$\Leftrightarrow$

$p_1^x, \dots p_{g-1}^x$ form unique hyperplane.

$B \subseteq (\mathbb{P}^{g-1})^*$ the branching locus $\mathcal{G}_\Theta$.

Lemma

Let $C^* \subset (\mathbb{P}^{g-1})^*$ the dual of the curve, then $\overline{B} = C^*$

K3 surfaces and other generalizations

Hodge theory for complex varieties

$$H^r(X;\mathbb{C}) \cong H^r(X;\mathbb{Z}) \otimes \mathbb{C}$$

Theorem (The Hodge decomposition)

Let X be a smooth complex projective variety of dimension d. Then the complex cohomology decomposes as

$$H^r(X;\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X),$$

for each $0 \leq r \leq 2d$, where $H^{p,q}(X) = H^q(X, \Omega^p)$.

$$\begin{array}{ccc} & H^{1,1} & \\ H^{1,0} & & H^{0,1} \\ & H^{0,0} & \end{array}$$

$$\begin{array}{ccccc} & & H^{2,2} & & \\ & H^{2,1} & & H^{1,2} & \\ H^{2,0} & & H^{1,1} & & H^{0,2} \\ & H^{1,0} & & H^{0,1} & \\ & & H^{0,0} & & \end{array}$$

Hodge-Riemann bilinear relations

$$Q\colon H_k(X;\mathbb{Z}) \otimes H_k(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\rightsquigarrow$$

$$Q\colon H_k(X;\mathbb{C}) \otimes H_k(X;\mathbb{C}) \to \mathbb{C}$$

The Hodge decomposition can be used to show that these must satisfy the Hodge-Riemann bilinear relations:

$$\begin{aligned} (-1)^k Q(\phi,\psi) &= Q(\psi,\phi) \\ Q(H^{p,q},H^{r,s}) &= 0 && \text{if } p \neq r \text{ or } q \neq s \\ i^{p-q} Q(\psi,\overline{\psi}) &> 0 && \psi \in H^{p,q} \end{aligned}$$

What is a Torelli-type theorem?

Given a smooth complex projective variety X of dimension d

- a lattice $H^k(X;\mathbb{Z})$,
- a decomposition $H^k(X;\mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$, and
- a non-degenerate bilinear form $Q\colon H_k(X;\mathbb{Z}) \otimes H_k(X;\mathbb{Z}) \to \mathbb{Z}$, extending to $H^k(X;\mathbb{C})$, satisfying the Hodge-Riemann bilinear relations.

Together, this data is called a polarized Hodge structure.

Torelli-type theorem

A Torelli-type theorem is a theorem characterizing a variety in terms of its polarized Hodge structures.

What is a K3 surface?

A smooth complex algebraic surface X with $h^0(X, \Omega^2) = 1$ and $h^1(X, \mathcal{O}) = 0$ is a K3 surface.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

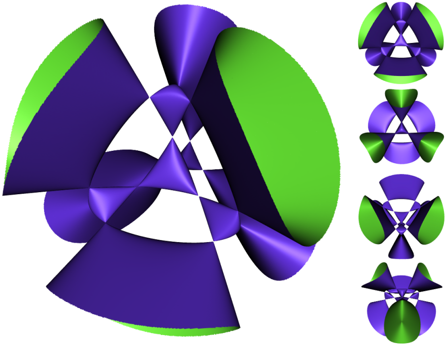


Figure: Kummer surface, (C. Rocchini CC BY-SA 3.0)

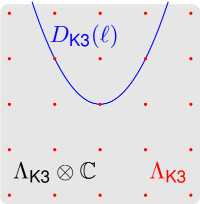
A *polarization* is a choice of an ample line bundle class $c \in H^{1,1} \cap H^2_{\mathbb{Z}}$.

Global Torelli for marked polarized K3's

The K3 lattice is $\Lambda_{\mathrm{K3}} \cong E_8^{\oplus 2} \oplus H^{\oplus 3}$, to which all $H_2(X;\mathbb{Z})$ are isometric.

A marking of a polarized K3 surface (X, c) is:

- isometry $\phi\colon H_2(X;\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$, and
- class $\ell = \phi(c) \in \Lambda_{\mathrm{K3}}$.



Hodge decomposition $\leftrightarrow \langle \omega \rangle = H^{2,0}(X)$

$\rightsquigarrow$ period point $\Phi(X)$, i.e. value of $\phi(\omega)$ in

$$D_{\mathrm{K3}}(\ell) \coloneqq \{[v] \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \mid \langle v, v \rangle = 0,\ \langle v, \overline{v} \rangle > 0,\ \langle v, \ell \rangle = 0\},$$

Theorem (Pjateckiĭ-Šapiro Šafarevič (1972))

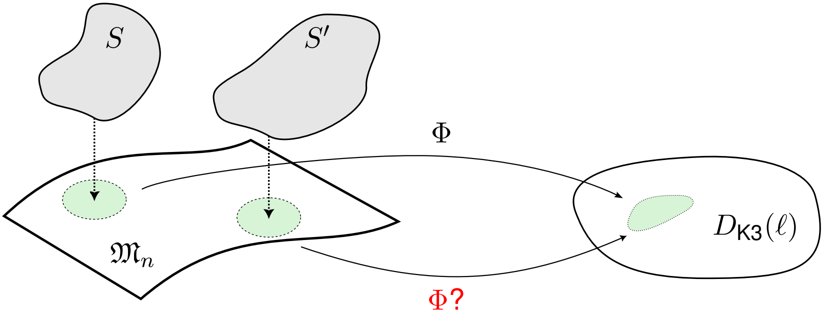
A marked polarized $K3$ surface X is uniquely determined by its period point $\Phi(X)$.

Moduli space of marked K3's

For each $n \geq 3$, there is a moduli space $\mathfrak{M}_n$ of marked polarized K3 surfaces.

Proposition

The period map $\Phi\colon \mathfrak{M}_n \to D(l)$ sending a marked K3 surface to its polarized Hodge structure is a local isomorphism.



Kummer surfaces

For A an abelian surface, take quotient by action $\sigma : x \mapsto -x$.

Definition (Kummer surface)

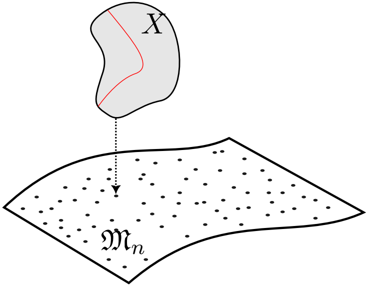
The resolution X of A/σ is a K3 surface, called a Kummer surface. It is special if A contains an elliptic curve.

Step 1: A K3 surface X is a special Kummer surface iff $\exists\, a \in \mathrm{Pic}(X)$ with $a^2 = 0$ and a induces a particular lattice.

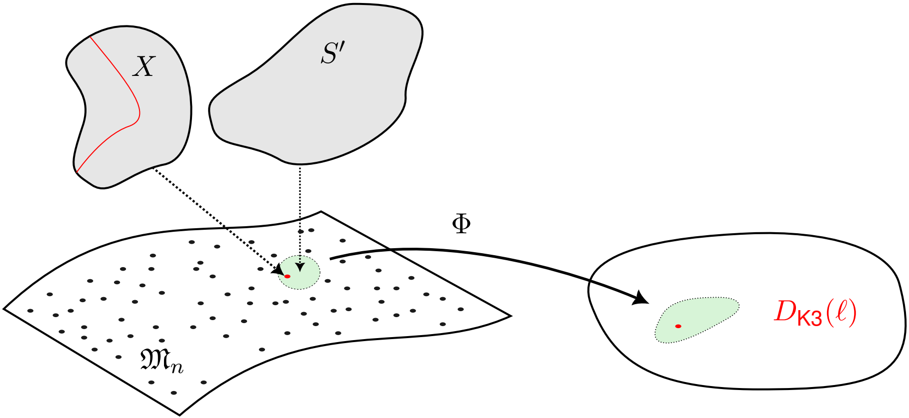
$\rightsquigarrow$

For special Kummer surfaces, $\Phi(X) = \Phi(X') \implies X \cong X'$.

Step 2: Special Kummer surfaces are dense



Finishing the proof



Global Torelli for threefolds and fourfolds

The third row of the Hodge diamond of a cubic threefold X has the form:

$$0 \quad H^{2,1} \quad H^{1,2} \quad 0.$$

Then $J(X) = H^{2,1}(X)/H_3(X;\mathbb{Z})$ is a principally polarized abelian variety, the 2nd intermediate Jacobian.

Theorem (Tyurin (1971), Clemens-Griffiths (1972))

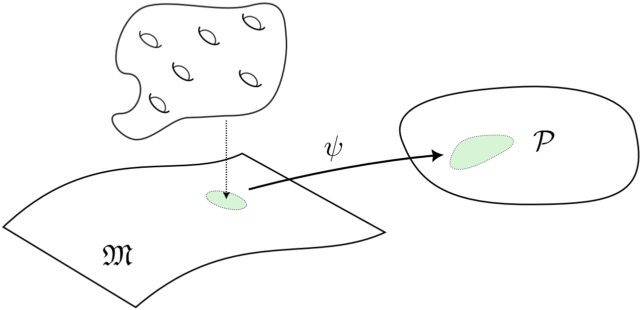
A non-singular cubic threefold is uniquely determined by its polarized intermediate Jacobian.

Theorem (Voisin (1986))

Two smooth cubic fourfolds X and X' are isomorphic if and only if there exists an isometry $H^4(X;\mathbb{Z}) \to H^4(X';\mathbb{Z})$ preserving the primitive classes and Hodge structure.

Global, Local, & Weak Torelli

For $\mathfrak{M}$ the moduli space of polarized algebraic varieties (X, ω), $\mathcal{P}$ the classifying space of polarized Hodge structures associated to (X, ω).



1. **Global Torelli**: $\psi\colon \mathfrak{M} \to \mathcal{P}$ is an embedding.
2. **Local Torelli at** (X, ω): differential $d\psi\colon T_{[X]} \to T_{\psi[X]}$ is injective.
3. **Weak Torelli** $\exists\, \mathfrak{M}' \subseteq \mathfrak{M}$ so that $\psi|_{\mathfrak{M}'}$ is an embedding.

References I

Torelli R.,
'Sulle varietá di Jacobi',
Rend. Accad. Lincei, 22(5):98–103, 1914.

Andreotti, A.,
'On a theorem of Torelli',
Amer. J. Math., 80:801–828, 1958.

Pjateckiĭ-Šapiro, I. I. and Šafarevič, I. R.,
'Torelli's theorem for algebraic surfaces of type K3',
Izv. Akad. Nauk SSSR Ser. Mat., 35:530–572, 1971.

Burns, D. and Rapoport, M.
'On the Torelli problem for kählerian K3 surfaces',
Ann. Sci. École Norm. Sup. (4), 8:235–273, 1975.

References II

Clemens, H. C. and Griffiths, P. A.,
'Torelli's theorem for algebraic surfaces of type K3',
Ann. of Math. (2), 95:281–356, 1972.

Voisin, C.,
'Théorème de Torelli pour les cubiques de $\mathbb{P}^5$',
Invent. Math., 3:577–601, 1986.

Barth, W., Peters, C. and Van de Ven, A.
Compact complex surfaces. ,
Springer-Verlag, 1984.

References III

- Kulikov, Vik. S. and Kurchanov, P. F.
  Complex algebraic varieties: periods of integrals and Hodge structures, Algebraic geometry III,
  Encyclopaedia Math. Sci., Springer, 1998.
- Griffiths, P. and Harris, J.
  Principles of algebraic geometry,
  Wiley-Interscience, 1978.

UiO : Department of Mathematics
University of Oslo

Edvard Aksnes

Torelli theorems

Trial Lecture