UiO : Department of Mathematics
University of Oslo

Tropical homology manifolds

Public Defense of doctoral thesis

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March 7, 2024

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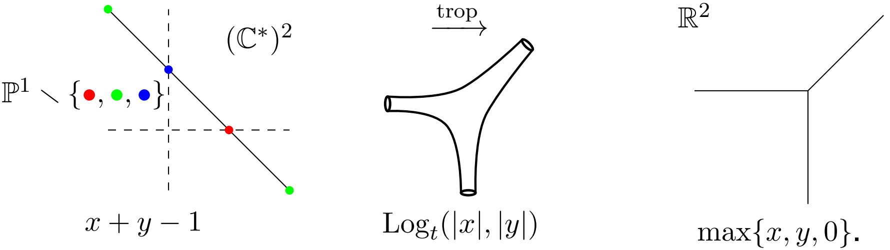
Background and preliminaries

Tropical Geometry

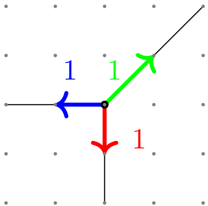
Philosophy: Form a bridge between *algebraic geometry* and *combinatorics*.



- Recovering invariants, e.g. dimension, degree,
- Correspondence theorems, e.g. Gromov-Witten, Welschinger,
- Geometric models for combinatorial objects, e.g. matroids,
- ... and study the geometry of tropical varieties.

Tropical varieties

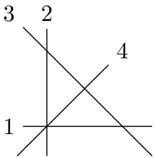
A tropical variety is locally a *balanced* rational fan.

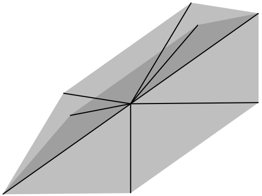


Balancing condition:

$$1 \leftarrow + 1 \downarrow + 1 \nearrow = 0$$

Bergman fans of matroids are tropical linear spaces, and the local models of *tropical manifolds*.





$M \sim \{124, 13, 23, 34, 1, 2, 3, 4\} \xrightarrow{\text{trop}} \Sigma_M$

Tropical Cohomology

Itenberg, Katzarkov, Mikhalkin and Zharkov define tropical (co)homology

$$H_\bullet^{\text{trop}}(X) := \bigoplus_{p+q=\bullet} H_{p,q}^{\text{trop}}(X), \qquad H^\bullet_{\text{trop}}(X) := \bigoplus_{p+q=\bullet} H^{p,q}_{\text{trop}}(X).$$

Theorem (IKMZ)

Let $\pi\colon \mathfrak{X} \to \mathbf{D}^*$ be smooth family of complex projective varieties, tropicalizing to a tropical manifold X. Then the $\dim H^{p,q}_{\text{trop}}$ are equal to Hodge numbers of the generic fibres.

Flavor example: $\dim H^1(\mathbb{P}^1 \setminus \{\bullet, \bullet, \bullet\}) = 2 = \dim H^{0,1}_{\text{trop}}(\prec)$

Poincaré duality in tropical geometry

Balancing condition $\rightsquigarrow$ fundamental class $[X] \in H^{\mathrm{trop}}_{d,d}(X)$

Theorem (Jell, Shaw, Smacka, Rau)

The Bergman fan Σ_M of a matroid M has isomorphisms:

$$\frown [\Sigma_M] \colon H^{p,q}_{\mathrm{trop}}(\Sigma_M) \to H^{\mathrm{trop}}_{d-p,d-q}(\Sigma_M),$$

for all $0 \leq p, q \leq d := \dim X$. This is called tropical Poincaré duality.

Contents of the thesis

Three papers

1. Tropical Poincaré duality spaces
2. Cohomologically tropical varieties
3. Transverse curve arrangements, cohomologically tropical arroids and maximal varieties.

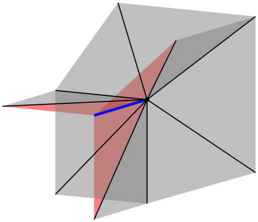
Tropical Poincaré duality spaces

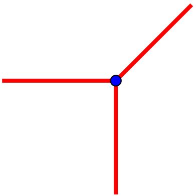
Tropical homology manifolds

$$\frown [\Sigma] \colon H^{p,q}_{\mathrm{trop}}(\Sigma) \to H^{\mathrm{trop}}_{d-p,d-q}(\Sigma), \quad 0 \le p, q \le d \coloneqq \dim(\Sigma)$$

Definition (A.)

A tropical fan is a tropical homology manifold if all its star fans Σ^σ satisfy tropical Poincaré duality.





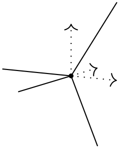
Locally matroidal spaces are tropical homology manifolds.

Classifying tropical homology manifolds

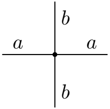
Theorem (A.)

A one-dimensional tropical fan is a tropical homology manifold if and only if it is uniquely balanced.

- Codim one: Newton polytope must be simplex.
- Euler characteristic comparison criteria.



Uniquely balanced



Not uniquely balanced.

Theorem (A.)

Let Σ be a tropical fan of dimension $d \geq 2$, with only top tropical homology. If all proper stars satisfy TPD, then Σ satisfies TPD.

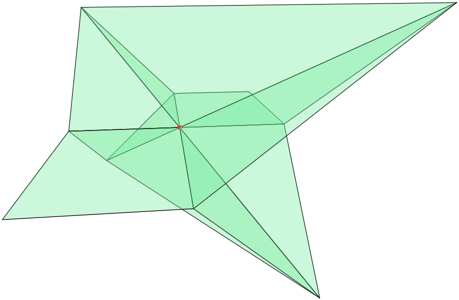
Proof idea

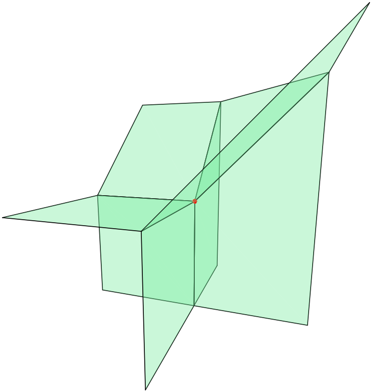
Induction on dimension:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{0,p}(\Sigma) & \longrightarrow & \bigoplus_{\tau\in\Sigma^1} H^{0,p}(\Sigma^\tau) & \longrightarrow & \bigoplus_{\sigma\in\Sigma^2} H^{0,p}(\Sigma^\sigma) & \longrightarrow \cdots \\
& \Big\downarrow \frown[\Sigma] & & \Big\downarrow \oplus_\tau \frown[\Sigma^\tau] & & \Big\downarrow \oplus_\sigma \frown[\Sigma^\sigma] & \\
0 \longrightarrow & H^{BM}_{d,d-p}(\Sigma) & \longrightarrow & \bigoplus_{\tau\in\Sigma^1} H^{BM}_{d,d-p}(\Sigma^\tau) & \longrightarrow & \bigoplus_{\sigma\in\Sigma^2} H^{BM}_{d,d-p}(\Sigma^\sigma) & \longrightarrow \cdots
\end{array}$$

Take resolution of lower complex into double complex, study spectral sequence, prove exactness.

Non-TPD fans and open directions



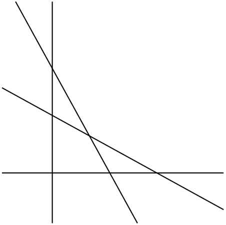


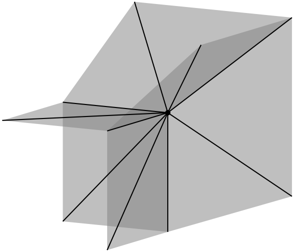
- What is the geometry behind the homology vanishing condition?
- Tropical Poincaré duality $\implies$ tropical homology manifold?

Cohomologically tropical varieties
(w/ Amini, Piquerez, & Shaw)

Tropical and singular cohomology

When is tropical cohomology *isomorphic* to singular cohomology?





$\mathbf{X}_{\mathcal{L}} := \mathbb{P}^2 \setminus \cup_{\mathbf{L}} \mathbf{L} \quad \xrightarrow{\text{trop}} \quad \Sigma_{M(\mathcal{L})} = \text{trop}(\mathbf{X}_{\mathcal{L}})$

Zharkov (2013) shows: $H^\bullet(\mathbf{X}_{\mathcal{L}}) \cong H^\bullet_{\text{trop}}(\Sigma_{M(\mathcal{L})})$

In general: Isomorphism using which map?

Tropical compactifications

$$\mathbf{X} \subseteq \mathbf{T} \rightsquigarrow X := \mathrm{trop}(\mathbf{X}) \subseteq \mathbb{R}^n$$

Choose a unimodular fan structure Σ on X:

Complex toric variety	**Tropical toric variety**
$\Sigma \rightsquigarrow \mathbb{CP}_\Sigma$	$\Sigma \rightsquigarrow \mathbb{TP}_\Sigma$
$\mathbf{X} \subset \overline{\mathbf{X}} \subseteq \mathbb{CP}_\Sigma$	$X \subset \overline{X} \subseteq \mathbb{TP}_\Sigma$
$\sigma \in \Sigma \rightsquigarrow$ varieties $\mathbf{X}^\sigma \subset \overline{\mathbf{X}}^\sigma$	$\sigma \in \Sigma \rightsquigarrow$ tropical fans $X^\sigma \subset \overline{X}^\sigma$

For schön varieties, $\overline{\mathbf{X}} \setminus \mathbf{X}$ is an snc divisor, for *any* choice of Σ.

Idea: Inductively relate all $H^\bullet(\overline{\mathbf{X}}^\sigma)$ and $H^\bullet_{\mathrm{trop}}(\overline{X}^\sigma)$.

Cohomologically tropical varieties

$$\tau^*: \quad \begin{array}{ccccc} H^{2\bullet}_{\mathrm{trop}}(\overline{X}) & & \longrightarrow & & H^{2\bullet}(\overline{\mathbf{X}}) \\ \pi\downarrow & & & & \uparrow i^* \\ \bigoplus_k H^{k,k}_{\mathrm{trop}}(\overline{\mathbf{X}}) & \xrightarrow{\cong} & A^\bullet(\Sigma) & \xrightarrow{\cong} & H^\bullet(\mathbb{CP}_\Sigma) \end{array}$$

Definition

The variety $\mathbf{X}$ is cohomologically tropical if $\tau^*: H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ is an isomorphism for each $\sigma \in \Sigma$.

Theorem (A., Amini, Piquerez, Shaw)

Let $\mathbf{X} \subseteq \mathbf{T}$ be a schön variety with $X = \mathrm{trop}(\mathbf{X})$. Then TFAE:

- *$\mathbf{X}$ is cohomologically tropical.*
- *$\mathbf{X}$ is wunderschön and X is a tropical homology manifold,*

Moreover, if either statement holds, X is Kähler.

Characterization of CT varieties

Definition

The variety $\mathbf{X}$ is wunderschön if for all cones $\sigma \in \Sigma$,

- $\mathbf{X}^\sigma$ is smooth and connected, and
- the *mixed Hodge structure* on the cohomology $H^k(\mathbf{X}^\sigma)$ is pure of degree $2k$ for all k.

Proof idea: Compare the weight spectral sequence for mixed Hodge structure of $(\overline{\mathbf{X}}, \mathbf{X})$:

$$0 \longrightarrow H^k(\mathbf{X}) \longrightarrow \bigoplus_{\sigma\in\Sigma_k} H^0(\overline{\mathbf{X}}^\sigma) \longrightarrow \cdots \longrightarrow \bigoplus_{\zeta\in\Sigma_1} H^{2k-2}(\overline{\mathbf{X}}^\zeta) \longrightarrow H^{2k}(\overline{\mathbf{X}}) \longrightarrow 0$$

with the *tropical Deligne exact sequence*:

$$0 \longrightarrow H^k(X) \longrightarrow \bigoplus_{\sigma\in\Sigma_k} H^0(\overline{X}^\sigma) \longrightarrow \cdots \longrightarrow \bigoplus_{\zeta\in\Sigma_1} H^{2k-2}(\overline{X}^\zeta) \longrightarrow H^{2k}(\overline{X}) \longrightarrow 0$$

Questions and directions

In dimension one:

Wunderschön $\Leftrightarrow \mathbb{CP}^1 \setminus \{\text{points}\}$

Tropical homology manifold $\Leftrightarrow$ Uniquely balanced

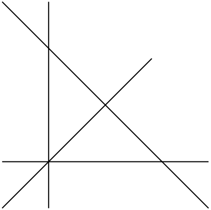
More broadly:

Which varieties are wunderschön/cohomologically tropical, in dimension two or higher?

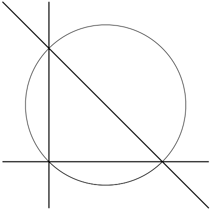
Curve arrangements and arroids

From line to curve arrangements

Question: How can we axiomatize the properties of an arrangement of curves in $\mathbb{P}^2$, and recover its tropicalization and cohomology?



$\rightsquigarrow$ Matroid



$\rightsquigarrow$ **?**

$\mathrm{trop}(\text{Complement}) = \text{Bergman fan of matroid}$

The arroid of a curve arrangement

Definition (A.)

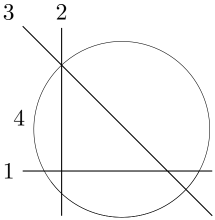
An arroid $\mathbf{A}$ is the following data:

1. Set of *divisors* $E := \{1, \ldots, n\}$,
2. *Degree function* $d \colon E \to \mathbb{N}$.
3. *Points* multiset $\mathcal{P}$ of subsets of E.

The points $\mathbf{p} \in \mathcal{P}$ have *multiplicities*

$$m_{\mathbf{p}} \colon \mathbf{p} \times \mathbf{p} \to \mathbb{Z},$$

which must satisfy a *Bézout condition*.



$$E = \{1, 2, 3, 4\}$$
$$d(4) = 2$$
$$\mathcal{P} = \{14, 14, 234, 24, 34\}$$

Theorem (A.)

For $\mathbf{A}$ a transversal arroid, the arroid fan $\Sigma_{\mathbf{A}}$ is a balanced fan.

Some arroid fan results

Theorem (A.)

Let $\mathcal{B}$ be a transverse arrangement of curves in $\mathbb{P}^2_K$, then

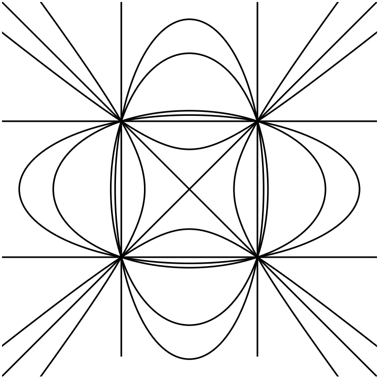
$$\operatorname{trop}(\mathbf{X}_{\mathcal{B}}) = \Sigma_{\mathbf{A}_{\mathcal{B}}}.$$

Theorem (A.)

For $\mathbf{X}_{\mathcal{B}}$ the complement of a simple line & conic arrangement $\mathcal{B}$, then

$\mathbf{X}_{\mathcal{B}}$ *cohomologically tropical* $\Leftrightarrow$ $\Sigma_{\mathbf{A}_{\mathcal{B}}}$ *uniquely balanced along each of its rays.*

Further directions



- Going beyond the transversal case.
- Combinatorics of hypersurface intersections?

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