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Birational Invariants and the Lüroth problem

Rationality properties and invariants to distinguish them

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Birational Geometry

Definition

Let X, Y be varieties. We say X, Y are *birational* if there exists opens U, V such that

$$X \supseteq U \simeq V \subseteq Y$$

Goal of birational geometry:

- Classify varieties up to birational equivalence
- Classify function fields of varieties

Lüroth's theorem

Theorem

Let K be a field and M an intermediate field between K and $K(X)$,

$$K \subseteq M \subseteq K(X)$$

Then there exists a rational function $f(X) \in K(X)$ such that

$$M = K(f(X)).$$

Remark

Geometrically, if C is a curve with a dominant rational map $\mathbb{P}^1 \dashrightarrow C$, then C is rational.

Proof of Lüroth's theorem (over $\mathbb{C}$)

$$
\begin{array}{ccc}
\mathbb{C} & \subset & \mathbb{P}^1 := \mathbb{C} \cup \infty \\
\downarrow & & \downarrow f \\
C \setminus Sing(C) & \subset & \overline{C}
\end{array}
$$

The projective line has no holomorphic forms, so $f^*\omega = 0$ $\implies \omega = 0$, so Riemann shows that $\overline{C} \simeq \mathbb{P}^1$.

Remark

There are entirely algebraic proofs, valid over any field.

Lüroth's Theorem in dimension 2

Theorem (Enriques, Castelnuovo)

Let X be a smooth, complex, surface. Assume there is a dominant rational map $\mathbb{P}^2 \dashrightarrow X$, then X is rational.

Proof.

Similar to the proof of the Lüroth problem.

$$\mathbb{P}^2 \dashrightarrow X \text{ dominant} \implies \text{no 1- or 2-forms } \text{"}\!\implies\!\text{"} \text{ rational}$$

■

Remark

The Enriques surface has no holomorphic forms, but is non-rational.

Higher dimensions

- It was suspected Lüroth's theorem did not extend to higher dimensions
- Focus on cubic- and (2,3)-complete intersection threefolds by Fano, Enriques and others
- Several erroneous results published
- First counterexamples to a Lüroth type result came in 1971

Increasingly Irrational

Let X be a projective variety of defined over an algebraically closed field k. We say X is:

1. *rational* if X is birational to $\mathbb{P}^n$
2. *stably rational* if $X \times \mathbb{P}^k$ is birational to $\mathbb{P}^{n+k}$
3. *unirational* if there is a dominant map $\mathbb{P}^n \dashrightarrow X$
4. *rationally connected* if any two general points can be connected by a rational curve
5. *uniruled* if there is a dominant map $\mathbb{P}^1 \times Y \to X$, with $\dim Y = \dim X - 1$

Invariants

- Holomorphic forms
- The birational automorphism group $Bir(X)$
- Topological invariants
- Algebraic cycles and Hodge Theory

Holomorphic forms

When X is smooth we can consider the holomorphic forms on X.

Theorem

For any $k \geq 0$ the space $H^0(X, \Omega_{X/k}^{\otimes k})$ is a stable birational invariant.

Proof.

Check that it is invariant when replacing X with $X \times \mathbb{P}^n$. By smoothness any birational map is defined outside a codimension 2 subset Z. If $U = X \setminus Z$, we have an injective map

$$\varphi^* : H^0(Y, \Omega_{Y/K}^{\otimes k}) \to H^0(U, \Omega_{U/K}^k).$$

By normality, $H^0(U, \Omega_{U/K}^k) \simeq H^0(X, \Omega_{X/K}^k)$ ∎

However, rationally connected $\implies$ no forms

Increasingly Irrational

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Remark

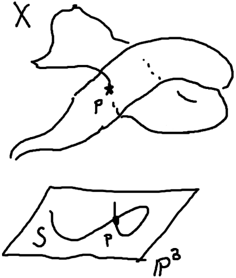
The implications upwards do not hold.
(Unknown if rationally connected $\implies$ unirational)

Unirational non-stably-rational varieties

- Let $X \to \mathbb{P}^3$ be a quartic double solid ramified along S
- Let p be an ordinary double point of S
- Let the projection of S from p be ramified along two transversal cubics

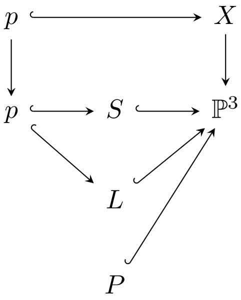
Proposition

If $\widetilde{X}$ is the blowup of X at its singular points, then $\widetilde{X}$ is unirational.



Artin and Mumfords Example is unirational

- Pick a node p on S (and X)
- A line L meets S at (p, x, y)
- The inverse image of L in X is a conic
- The inverse image of the plane P is a Del Pezzo surface $\Sigma \subset X$
- So X is a conic bundle with a rational multisection
- $\Sigma \times_{\mathbb{P}^2} X \to X$ shows unirationality



Artin-Mumford invariant

Lemma

Let X be a complex variety. The group $H^3_B(X,\mathbb{Z})_{tors}$, the torsion subgroup of the i-th Betti cohomology of X^{an}, is a stable birational invariant.

Proof.

$$H^3_B(X\times\mathbb{P}^1,\mathbb{Z}) = H^3_B(X,\mathbb{Z})\oplus H^1(X,\mathbb{Z})$$

and if $\tau\colon \widetilde{X}\to X$ is a blow-up in a smooth center $Z\subset X$:

$$H^3_B(\widetilde{X},\mathbb{Z}) = H^3_B(X,\mathbb{Z})\oplus H^1_B(Z,\mathbb{Z})$$

There is never torsion in $H^1_B(Y,\mathbb{Z})$ ■

Unirational non-stably-rational varieties

Theorem (Artin-Mumford)

If $\widetilde{X}$ is the blowup of X at its singular points, then $\widetilde{X}$ has non-trivial torsion in $H^3_B(\widetilde{X}, \mathbb{Z})$. In particular, it is not stably rational.

Proof.

Artin and Mumford prove this by explicitly computing the cohomology and constructing a non-zero 2-torsion class. ∎

Intermediate Jacobian

Definition

Let X be a smooth complex threefold. The intermediate Jacobian $J^3(X)$ is defined as:

$$J^3(X) := H^3_B(X, \mathbb{C})/(H^{3,0}(X, \mathbb{C}) \oplus H^3(X, \mathbb{Z})/Torsion)$$

This is a complex torus, and if $(H^{3,0}(X, \mathbb{C}) = 0$, $J^3(X)$ is a principally polarized abelian variety.)

Clemens-Griffiths Criterion

Theorem

Let X be a smooth complex projective threefold with

$$H^{3,0}(X) = H^{1,0}(X) = 0.$$

If X is rational, then (J^3, θ_X) is a direct product of Jacobians of smooth curves.

Proof.

A birational map $\mathbb{P}^3 \dashrightarrow X$ factors through blow-ups in smooth centers:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle\tau} & & \\ \mathbb{P}^3 & & \end{array}$$

One can compute that $J^3(Y) = \prod (J(C_i), \theta_{C_i})$, where C_i are the centers of blow-ups. ∎

Stably rational, non-rational varieties

Theorem (BCTSSD '85)

Let $P(x,t) = x^3 + p(t)x + q(t)$ be an irreducible polynomial in $\mathbb{C}[x,t]$, whose discriminant $\delta(t) := 4p(t)^3 + 27q(t)^2$ has degree ≥ 5. The affine hypersurface $V \subset \mathbb{C}^4$ definied by $y^2 - \delta(t)z^2 = P(x,t)$ is stably rational but not rational.

Specifically $V \times \mathbb{P}^3$ is rational (later improved to $V \times \mathbb{P}^2$ is rational by Shepherd-Barron)
Irrationality is proven by using the intermediate Jacobian

Birational rigidity

Theorem (Iskovskikh-Manin '71)

Let X be a smooth quartic threefold over a field of characteristic 0. Then the birational automorphism group $Bir(X)$ is finite

Corollary

A smooth quartic threefold is not rational.

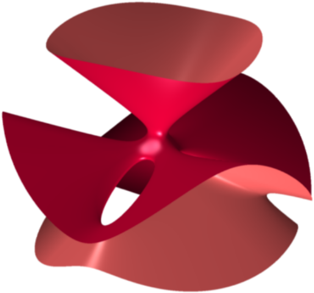
Hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d.

- $d = 1$ then $X \simeq \mathbb{P}^n$
- $d = 2$ then projection from a point shows X is rational
- $d \geq n + 1$, then $h^n(\Omega_{X/k}) \geq 1$ so X is not (stably) rational
- $3 \leq d \leq n$ are very difficult cases

Cubic Surfaces

- Rational if it contains two (Galois invariant) disjoint lines (in particular over $\mathbb{C}$)
- Unirational if it contains a point [Segre,Kollár]



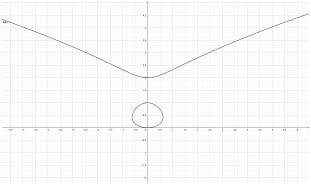
Cubic surfaces

Example

The cubic surface $X \subset \mathbb{P}^3_{\mathbb{R}}$ defined by $x^2 + y^2 = f_3(z)$, when f_3 has three distinct roots, is unirational but not rational.

Proof.

X has two disjoint components. ∎



Cubic threefold

Proposition

Let $X \subset \mathbb{P}^4$ be a smooth complex cubic threefold. Then X is unirational.

Proof.

Blow up a line to get a conic bundle structure $\widetilde{X} \to \mathbb{P}^2$. The exceptional divisor E is a rational multisection of degree 2. The fiber product

$$E \times_{\mathbb{P}^2} \widetilde{X} \to X$$

is rational, and shows that $\widetilde{X}$ is unirational. ■

Non-rationality of the cubic threefold

Theorem (Clemens-Griffiths ’72)

Let $X \subset \mathbb{P}^4$ be a smooth complex cubic threefold. Then X is non-rational.

Proof.

- Use the intermediate Jacobian and the Clemens-Griffiths criterion.
- $J^3(X)$ is an irreducible PPAV of dimension 5
- The singular locus of its Theta divisor is a single point
- Riemann proves Theta divisors of Jacobians of curves have singular locus of codimension ≤ 4

∎

Cubic fourfold

- The general complex cubic fourfold is conjectured to be non-rational
- Smooth rational complex cubic fourfolds exist in codimension 1

Example

Let X be a smooth cubic fourfold containing two disjoint planes. Then X is rational

Higher dimensional cubics

Conjecture

The general complex cubic hypersurface is non-rational except in dimension 2

Question

Are there smooth rational cubic hypersurfaces in higher dimensions?

Recent developments

Lots of recent progress in finding non-stably rational varieties

Theorem (Voisin 2015)

A double cover branched along a very general quartic surface is not stably rational.

Based on *decomposition of the diagonal* and the Artin-Mumford example.

Solved the remaining complete intersection threefolds except the cubic threefold (for very general).

References

Good survey papers:

- Beauville 2016
- Voisin 2016
- Hassett 2016
- Pirutka 2018

Milestone papers:

- Iskovskih and Manin 1971
- Artin and Mumford 1972
- Clemens and Griffiths 1972
- Beauville et al. 1985
- **Voi15**

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