Toric Geometry

- A special class of varieties constructed from combinatorics
- A rich class of varieties: can characterize many geometric properties purely in terms of combinatorics.

Toric varieties form a rich class of examples in alg geo.

Goal: Construct an example of a (complete) proper variety that is not projective.

All varieties are over $\mathbb{C}$
(Sep. Schemes of finite type over $\mathbb{C}$) integral.

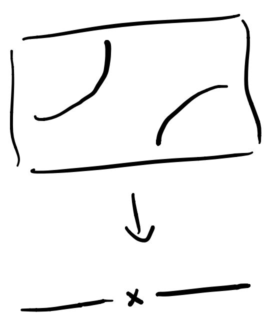
Definitions

1) A variety X is (complete) proper if for any other variety Y, the projection $X \times Y \to Y$ is closed.

Example: Any projective (scheme) variety is proper (not easy)

$\mathbb{A}^1$ is not proper.

$$\mathbb{A}^2 \simeq \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$



$X = \{xy=1\}$ in $\mathbb{A}^2$ is closed but maps to $\mathbb{A}^1 \setminus \{0\}$ which is open.

2) A proper variety X is projective iff it has an ample line bundle (equiv to embedding $X \hookrightarrow \mathbb{P}^n$)

We will later construct a variety which is proper but admits no ample line bundle.

Toric Geometry

Def

An n-dimensional (algebraic) torus is $\mathbb{G}_m^n$ where $\mathbb{G}_m$ is the multiplication group $\mathbb{C}^*$ with normal multiplication. i.e $\mathbb{G}_m^n = \underbrace{\mathbb{C}^* \times \mathbb{C}^* \times \dots \times \mathbb{C}^*}_{n\text{-times}}$.

Definition (1)

A toric variety is an irreducible variety X with a Zariski dense torus $\mathbb{G}_m^n \hookrightarrow X$ such that the action $\mathbb{G}_m^n \times \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^n$ extends to an action $\mathbb{G}_m^n \times X \to X$.

Ex: $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$ is toric. $\mathbb{C}^*$ embedded via $\mathbb{C}[t, t^{-1}] \leftarrow \mathbb{C}[t]$. Action $\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1$

$$(t, x) \longmapsto tx.$$

Definition (2)

An affine toric variety $X^{\subset \mathbb{A}^n}$ is a variety cut out by a prime ideal I generated by binomial equations. (Eq's of the form $x_{i_1} \cdots x_{i_r} = x_{j_1} \cdots x_{j_k}$.

Ex

$$\underset{X}{\{xy - zw = 0\}} \subset \mathbb{A}^4 \quad \text{is toric.}$$

The dense torus is given by $(x, y, z, \frac{xy}{z})$ for $x, y, z \neq 0$.

$$(\mathbb{C}^*)^3 \longrightarrow X$$
$$(x,y,z) \longmapsto (x, y, z, \frac{xy}{z})$$

Definition/Construction (3)

Let $M \simeq \mathbb{Z}^n$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = M^\vee$

$M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$

- A strongly convex rational polyhedral cone $\sigma \subset M_{\mathbb{R}}$ is a convex cone σ with apex at the origin such that
  - (rational) it is generated by finitely many vectors
  - (strong) it contains no line through the origin.

Define the dual cone $\sigma^\vee \subset M_{\mathbb{R}}^\vee = N_{\mathbb{R}}$ by

$$\sigma^\vee = \{ n \in N \mid \langle n, m \rangle = n(m) \geq 0 \text{ for all } m \in \sigma \}$$

$S_\sigma = \sigma^\vee \cap N$ is the semigroup of lattice points (under addition)

Gordans Lemma: S_σ is finitely generated.

Consequence: The algebra $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra.

Notation: we denote elements of $\mathbb{C}[S_\sigma]$ by χ^m where $m \in S_\sigma$. for $m, n \in S_\sigma$ we set $\chi^m \chi^n = \chi^{m+n}$. $\chi^0 = 1$.

$U_\sigma = \mathrm{Spec}\, \mathbb{C}[S_\sigma]$ is the affine toric variety corresponding to σ.

For $\tau \subset \sigma$ a face, we get a map $U_\tau \to U_\sigma$ which embeds U_τ as a distinguished open set of U_σ.

Def

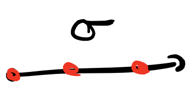
A fan $\Delta \subset M_{\mathbb{R}}$ is a collection of cones such that any two cones meets in a common face which is in the fan. Each cone σ in Δ gives a toric variety U_σ. For two cones σ_1, σ_2 & $\tau = \sigma_1 \cap \sigma_2$ we have

$$U_{\sigma_1} \leftarrow U_\tau \rightarrow U_{\sigma_2}$$

& we can glue along U_τ.

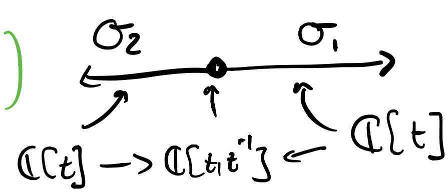
The resulting variety X_Δ is the <u>toric</u> variety corresponding to Δ.

<u>Ex</u>

1) 

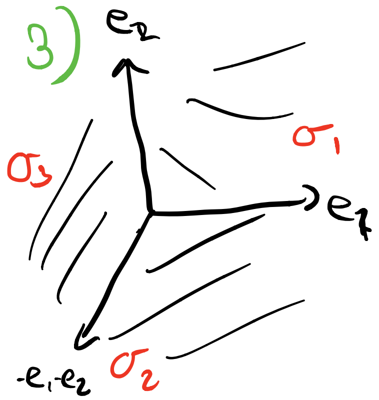
$S_\sigma = \mathbb{Z}_{\geq 0} \subset \mathbb{R}$ gen by 1

$\operatorname{Spec} \mathbb{C}[S_\sigma] \simeq \mathbb{A}^1$

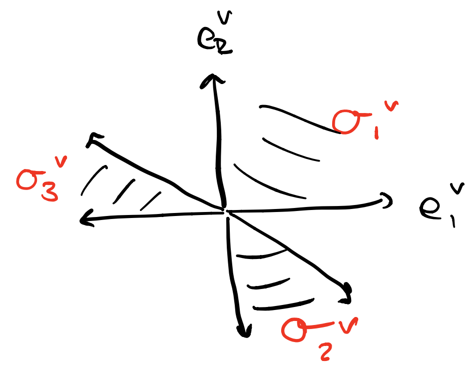
2) 

glue these $\rightsquigarrow X_\Delta = \mathbb{P}^1$.

3)



dual $\rightsquigarrow$



Each of these is an $\mathbb{A}^2$.

One checks that the gluing gives

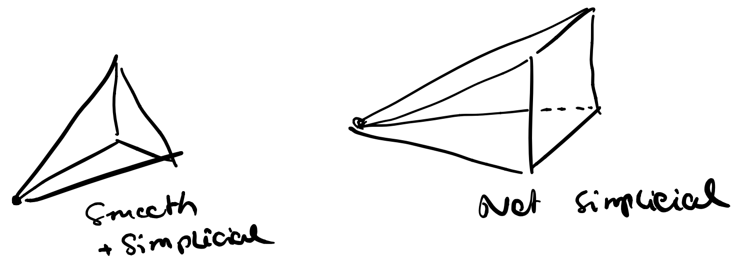
$X_\Delta = \mathbb{P}^2$.

Moral: The geometry of X_Δ is dictated by Δ.

Facts

Def A cone $\sigma \subset M_\mathbb{R}$ is said to be

1) Smooth if its minimal generators can be extended to a $\mathbb{Z}$-basis for M.
2) Simplicial if its minimal generators are linearly independent in $M_\mathbb{R}$.

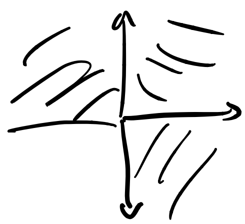


Def: A fan Δ is said to be

1) **Smooth** if all cones are smooth

2) **Simplicial** if all cones are simplicial

3) **Complete** if $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.



Not complete.

Facts

1) X_Δ is Smooth $\Leftrightarrow$ Δ smooth.

2) X_Δ has finite quotient sings $\Leftrightarrow$ Δ simplicial.

3) X_Δ Proper $\Leftrightarrow$ Δ complete.

$\mathbb{P}^2$. Complete (But also Proj)

Divisors

Recall:

- A Weil divisor D is a formal sum

$$D = \sum_{i \in I} a_i D_i , \quad |I| < \infty$$

where $a_i \in \mathbb{Z}$ & the D_i are closed integral subschemes of pure codim 1. (Prime divisor)

Any rational function $f \in K(X)$ induces a "Principal divisor" $\mathrm{div}(f) = \sum_{D \subset X} \mathrm{Val}_D(f) \, D$

Sum over all prime divisors.

D_1 & D_2 linearly equiv if $D_1 - D_2 = \mathrm{div}(f)$.

$Cl(X) = \text{Weil divisors} / \text{linear equiv.}$

- A cartier divisor on X is a closed subscheme D such that for any affine $\operatorname{Spec} A \subset X$

  $D \cap \operatorname{Spec} A = \operatorname{Spec} A/f$ for a non-zero divisor $f \in A$.

<u>Fact</u>

- If $\operatorname{Spec} \mathbb{C}[M]$ ($= U_\sigma$) is an affine toric variety then any T-invariant Cartier divisor $D = \sum a_\rho D_\rho$ is of the form

$$\operatorname{div}(\chi^{m_\sigma}) = \sum (m_\sigma, u_\rho) D_\rho$$

(u_ρ: minimal generator for ρ.)

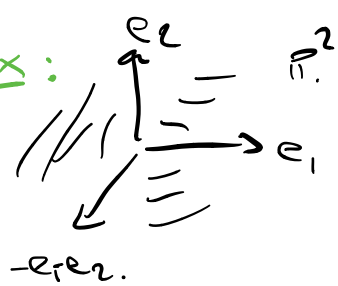
$$= D$$

(Note $(m_\sigma, u_\rho) = a_\rho$.)

Remark On a toric variety X, some divisors are invariant under the action of the torus.

$$\{T\text{-invariant Weil divisors}\} \overset{1:1}{\longleftrightarrow} \{\text{Orbit closures under } T\text{-action.}\}$$

Ex:

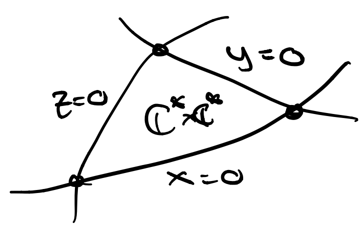


Action $(\mathbb{C}^*)^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$

$$(\lambda,\mu) \times [x:y:z] \longmapsto [\lambda x : \mu y : z]$$

Orbits:

- $x \neq 0, y \neq 0, z \neq 0$ — $(\mathbb{C}^*)^2$
- $x \neq 0, y \neq 0, z = 0$ — $\{z = 0\}$
- $x \neq 0$ $y = 0$ $z \neq 0$ — $\{y = 0\}$
- $x = 0$ $y \neq 0$ $z \neq 0$ — $\{x = 0\}$
- $x = y = 0$ $z \neq 0$ — $[0:0:1]$
- $x = z = 0$ $y \neq 0$ — $[0:1:0]$
- $x \neq 0$ $z = y = 0$ — $[1:0:0]$



$$\{\text{codim } 1 \text{ orbits}\} \overset{1:1}{\longleftrightarrow} \{\text{rays of the fan}\}.$$

We write for any Weil div on X_Δ

$$D = \sum a_\rho D_\rho \quad \text{for } \rho \text{ rays of } \Delta.$$

Def:

A <u>piecewise linear function</u> on a fan Δ is a continuous function $\varphi : |\Delta| \to \mathbb{R}$ that is linear on each $\sigma \in \Delta$. and $\varphi(|\Delta| \cap M) \subset \mathbb{Z}$.

It is <u>convex</u> if $\varphi(tu + (1-t)v) \geq t\varphi(u) + (1-t)\varphi(v)$. $\forall\, u, v \in |\Delta|$ & $t \in [0,1]$.

Ex:

Let $D = \sum a_\rho D_\rho$ be a Cartier divisor. On open toric affines U_σ $= \operatorname{Spec} \mathbb{C}[M]$ we have

$$D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma}) \quad \text{for some}$$

$m_\sigma \in M$, with $(m_\sigma, u_\rho) = -a_\rho)$

The support function $\varphi_D : |\Delta| \to \mathbb{R}$ of D is

$$\varphi_D : |\Delta| \to \mathbb{R}$$
$$u \longmapsto (m_\sigma, u) \text{ when } u \in \sigma.$$

Note: $D = -\sum \varphi_D(u_\rho) D_\rho$.

$$\left\{ \begin{array}{c} T\text{-invariant Cartier} \\ \text{divisors} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Piecewise linear} \\ \text{functions } \varphi : |\Delta| \to \mathbb{R} \end{array} \right\}$$

$$D \longmapsto \varphi_D$$

$$\sum \varphi(u_\rho) D_\rho \longleftarrow \varphi$$

Prop:

Let Δ be a fan with $|\Delta|$ convex and $\dim |\Delta| = n = \dim M_{\mathbb{R}}$.

1) A T-invariant Weil divisor iff it is given by a piecewise linear function as above.

· If D is T-Cartier with support function φ_D:

2) D is basept free $\Leftrightarrow$ φ_D is convex

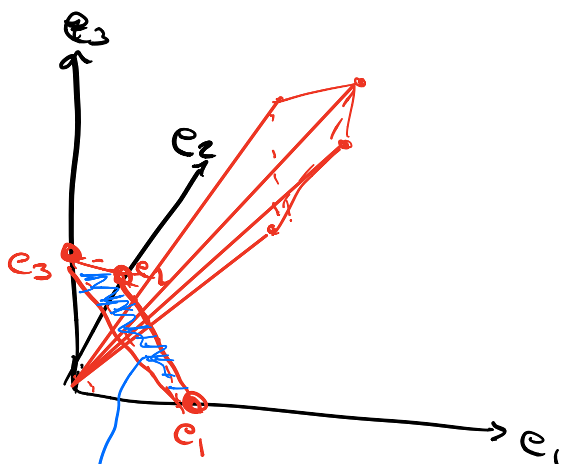
3) D is ample $\Leftrightarrow$ φ_D strictly convex.

Definition: $\varphi_D(u+v) > \varphi_D(u) + \varphi_D(v)$ for all $u, v \in |\Sigma|$ not in the same cone of Σ.

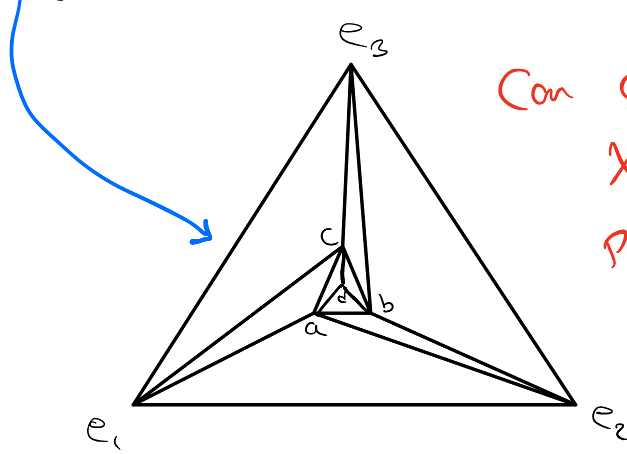
Main Example

Take the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{R}^3$, spanned by $\pm e_1, \pm e_2, \pm e_3$.
Further subdivide the $\mathbb{R}^3_{\geq 0}$ part of this fan by adding rays

$$a = (2,1,1),\ b = (1,2,1),\ c = (1,1,2)\quad d = (1,1,1).$$



Move a cone by filling in as in the following figure.



Can check:
X_Σ is smooth
$\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^7$.

Claim: No ample divisors !

Suppose $D = \sum a_\rho D_\rho$ is ample and let φ_D be the corresponding PLF. In particular

$$\varphi_D(e_1) = -a_{e_1}$$
$$\varphi_D(e_2) = -a_{e_2}$$
$$\varphi_D(e_3) = -a_{e_3}$$

By replacing D with $D + \mathrm{div}(\chi^{(-a_{e_1}, -a_{e_2}, -a_{e_3})})$ we can assume $\varphi_D(e_i) = 0 \quad i = 1, 2, 3.$

Note: $e_1 + b = (2,2,1) = e_2 + a$. e_1 & b are not in the same cone so by strict convexity

$$\varphi_D(e_1 + b) > \varphi_D(e_1) + \varphi_D(b) = \varphi_D(b)$$

e_2 & a are in the same cone so

$$\varphi_D(a) = \varphi_D(e_2) + \varphi_D(a) = \varphi_D(e_2 + a)$$
$$= \varphi_D(e_1 + b)$$
$$> \varphi_D(b)$$

So $\varphi_D(a) > \varphi_D(b)$.

Continue to get $\varphi_D(a) > \varphi_D(b) > \varphi_D(c) > \varphi_D(a)$

$\rightsquigarrow$ contradiction.