Degenerations & Stable Rationality.

- Birational invariants and examples.
- Grothendieck ring of varieties and the motivic volume.
- Strictly toroidal models and degenerations
- Degenerations of toric varieties.

DEF Smooth projective varieties X & Y over F are stably birational over F if $\exists\, n,m$ & a birational map $X \times \mathbb{P}^n_F \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^m_F$.
X is stably rational if it is stably birational to $\operatorname{Spec} F$.

Problem: Determine if a variety X is stably rational.

Attempt 1: Find a non-trivial stably birational invariant.

Let X be a smooth projective variety $/\mathbb{C}$.

1. Differential forms $H^0(X, \Omega_X^{\otimes k})$ $k>0$.
2. The fundamental group $\pi_1(X)$

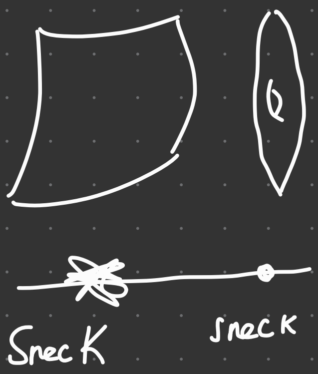
$\rightsquigarrow$ Trivial for rationally connected varieties (E.g Fano varieties)

3. $H^3(X, \mathbb{Z})_{torsion}$ (Can distinguish between unirational and rational)
   - Purely topological!
   - Can relate it to $Br(X)$, so also a bit algebraic.

The usual trick to find non-trivial classes.

4 Decomposition of the diagonal. (Condition in the chow group)

! This invariant specializes:



If $\mathcal{X} \to R = k[[t]]$ is a smooth scheme over R (flat + smooth fibers) & the generic fiber $\mathcal{X}_K$ has a decomposition of the diagonal, then the special fiber $\mathcal{X}_k$ also have one.

Recall

General: Complement of Zariski closed (equiv. Zariski open)

Very general: Complement of countably many Zariski closed.

(equiv. countable intersection of Zariski opens)

(HPT 16) family $\mathcal{X} \to C$ where $\mathcal{X}$ & C complex manifolds, C connected. fibers $\mathcal{X}_t$ are complex projective fourfolds.

- Very general fiber stably irrational.
- Dense (euclidean) set $U \subset C$ s.t $\mathcal{X}_t$ rational for all $t \in U$.

$\rightsquigarrow$ Not possibly to get rid of very general.

Examples

(1) Hypersurfaces $X_d \subset \mathbb{P}^n$ of degree d has canonical bundle $\omega_{X_d} = \mathcal{O}(-n-1+d)$. Stably irrational for $d \geq n+1$.

(in fact not even rationally connected)

(2) What about Fano varieties ($\omega_{X_d}^{-1}$ ample, i.e. $d \leq n$ above)?

deg \ dim	1	2	3	4	5	6	7
6	–	–	–	–	–	–	–
5	–	–	–	–	–	–	–
4	–	–	–	–	–	?	?
3	–	+	*	?	?	?	?
2	+	+	+	+	+	+	+
1	+	+	+	+	+	+	+

– Stably irrational

+ rational

* irrational.

• [Totaro 16] (Schreieder) $d \geq \log_2(n)+2 \Rightarrow$ Stably irrat.

• Ottem - Nicaise 19

Grothendieck ring & motivic volume

DEF The grothendieck ring of varieties over a field F is denoted $K_0(\mathrm{Var}_F)$ and generated by isomorphism classes of finite type F-schemes modulo relations

$$[X \setminus Z] = [X] - [Z]$$

for $Z \subset X$ closed subscheme. Denote by $\mathbb{L} = [\mathbb{A}^1_F]$ the class of $\mathbb{A}^1$.

Ring structure: $[X] \cdot [Y] = [X \times_F Y]$.

Remark This ring is complicated. If F is algebraically closed then there are always 0-divisors in the ring!

Example X finite type F-scheme. $Y \subset X$ closed subscheme. Then consider $Bl_Y X$.

$$Bl_Y X \setminus E \simeq X \setminus Y \quad \text{so} \quad [Bl_Y X \setminus E] = [X \setminus Y] \quad \text{hence}$$

$$[Bl_Y X] = [X] - [Y] + [E].$$

Example $\mathbb{P}^n \setminus \mathbb{P}^{n-1} \simeq \mathbb{A}^n$. So

$$[\mathbb{P}^n] = [\mathbb{P}^{n-1}] + \mathbb{L}^n = \mathbb{L}^n + \mathbb{L}^{n-1} + \dots + \mathbb{L} + 1 \qquad (1 = [\mathrm{Spec}\, F])$$

DEF Let $\mathbb{Z}[SB_F]$ be the free abelian group on stable birational equivalence classes. Ring structure $[X]_{sb} \cdot [Y]_{sb} = [X \times_F Y]_{sb}$.

Remark The above ring have no relations!! $[X]_{sb} = [Y]_{sb}$ if and only if X is stably birational to Y.

$\boxed{\text{Key theorem (Larsen-Lunts)}}$

Assume Char $F = 0$. Then there is a unique ring map

$$Sb : K_0(Var_F) \longrightarrow \mathbb{Z}[SB_F]$$

Such that for smooth and proper F-schemes X we have

$$Sb([X]) = [X]_{sb}$$

Sb is surjective with kernel generated by $\mathbb{L}$.

Corollary $K_0(Var_F)/(\mathbb{L}) \simeq \mathbb{Z}[SB_F]$.

Example Smooth and proper is important!

- $\mathbb{L} = [\mathbb{P}^1] - [\operatorname{Spec} F]$ so $Sb(\mathbb{L}) = Sb(\mathbb{P}^1) - Sb(\operatorname{Spec} F) = 0$
- $X \subset \mathbb{P}^2$ elliptic curve. $CX \subset \mathbb{P}^3$ cone over X. Blow up the vertex p. (then the exceptional divisor is the curve X).

$$\Rightarrow \quad [Bl_p\, CX] - [E] = [CX] - [\mathrm{Spec}\, F]$$

All these stably birational to X.

$$\Rightarrow \quad Sb(CX) = [\mathrm{Spec}\, F]_{sb} \neq [CX]$$ since CX is stably bir to X.

Models and degenerations

Let $R = k[[t]]$, $K = k((t))$ $\quad k = \bar{k}$

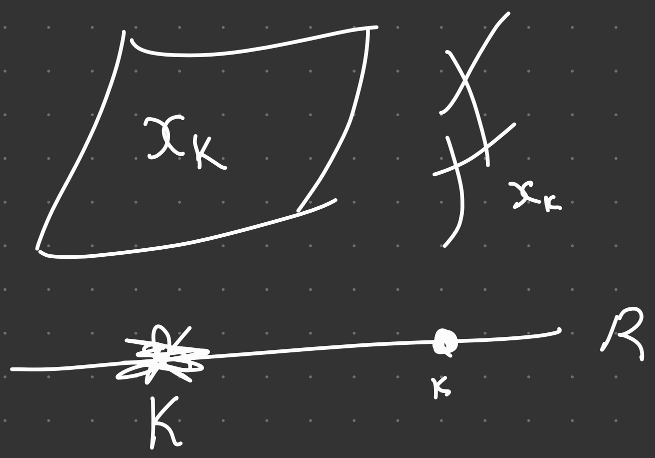
Algebraic closure of $K((t))$

$$R(\infty) = \bigcup_{n \geq 1} k[[t^{1/n}]] \qquad K(\infty) = \bigcup_{n \geq 1} k((t^{1/n})).$$

Note An R-scheme $\mathcal{X} \to R$ have two fibers.

Generic fiber: $\mathcal{X}_K$, a scheme over K

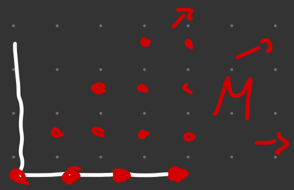
Special fiber: $\mathcal{X}_k$, a scheme over k.



$\boxed{\text{Goal}}$: Relate rationality of $\mathcal{X}_K$ to $\mathcal{X}_k$

DEF A monoid M is a toric monoid if it is the monoid of lattice points of a strictly convex rational polyhedral cone.

i.e. $\operatorname{Spec} k[M]$ is a toric variety.



DEF An $R(\infty)$-scheme $\mathfrak{X}$ is strictly toroidal if Zariski locally on $\mathfrak{X}$ there are smooth morphisms

$$\mathfrak{X} \longrightarrow \operatorname{Spec}\left(\frac{R(\infty)[M]}{(t^q - x^m)}\right)$$

where $q \in \mathbb{Q}$, $m \in M$, M is a toric monoid, and $\operatorname{Spec} \frac{R(\infty)[M]}{(x^m)}$ is reduced.

Intuition Think of $\mathfrak{X}$ as a scheme where the special fiber "looks" like a toric boundary. We allow singularities but they should be "close" to toric singularities.

Examples

(1) $\mathfrak{X}$ a regular R-scheme with smooth special fiber. Then

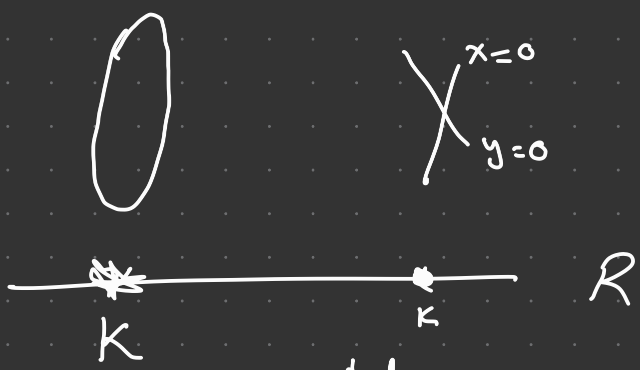
* $\mathfrak{X} \times_R R(\infty)$ is strictly toroidal.

(2) $\mathfrak{X}$ a regular R-scheme with strict normal crossing, then $\mathfrak{X} \times_R R(\infty)$ is strictly toroidal.

(Usualy called strictly semi-stable).

$\mathfrak{X} \subset \operatorname{Spec} R[x,y]$ defined by $t - xy$.

Generic fiber is a smooth conic and the special fiber is the union of two lines



DEF For $\mathfrak{X}$ strictly toroidal model with special fiber $\mathfrak{X}_k$, a <u>stratum</u> of $\mathfrak{X}_k$ is a connected component of an intersection of irreducible components in $\mathfrak{X}_k$. Let $S(\mathfrak{X})$ denote the set of strata.

Theorem (Nicaise-Shinder)

There is a unique ring map

$$\mathrm{Vol} : K_0(\mathrm{Var}_{K(\infty)}) \longrightarrow K_0(\mathrm{Var}_k)$$

Such that for every strictly toroidal model $\mathfrak{X}$ we have

$$\mathrm{Vol}(\mathfrak{X}_{K(\infty)}) = \sum_{E \in S(\mathfrak{X})} (-1)^{\operatorname{codim} E} [E].$$

Moreover we have an induced map

$$\mathbb{Z}[SB_{K(\infty)}] \xrightarrow{\mathrm{Vol}_{sb}} \mathbb{Z}[SB_k]$$

Such that $\boxed{\mathrm{Vol}_{sb}(\mathfrak{X}_{K(\infty)}) = \sum_{E \in S(\mathfrak{X})} (-1)^{\operatorname{codim} E} [E]_{sb}.}$

$$\begin{tikzcd} K_0(\mathrm{Var}_{K(\infty)}) \arrow[r, "\mathrm{Vol}"] \arrow[d, "sb"] \arrow[dr, phantom, "\circlearrowleft"] & K_0(\mathrm{Var}_k) \arrow[d, "sb"] \\ \mathbb{Z}[SB_{K(\infty)}] \arrow[r, "\mathrm{Vol}_{sb}"] & \mathbb{Z}[SB_k] \end{tikzcd}$$

Corollary: $\mathrm{Vol}_{sb}([\mathrm{Spec}\, K(\infty)]_{sb}) = [\mathrm{Spec}\, k]_{sb}$

$\Rightarrow$ If $\mathfrak{X}$ is strictly toroidal and

$$\sum_{E \in S(\mathfrak{X})} (-1)^{\mathrm{codim} E} [E]_{sb} \neq [\mathrm{Spec}\, k]_{sb}$$

then $\mathfrak{X}_K$ is <u>not</u> stably rational.

Example $\{f=0\}$ a quartic surface in $\mathbb{P}^3$. Let q_1, q_2 be very general quadric polynomials. Then

$$\mathfrak{X} = \{tf + q_1 q_2 = 0\} \subset \mathbb{P}^3_{R(\infty)} \quad \text{is strictly toroidal.}$$

$\mathfrak{X}_k = \{q_1 q_2 = 0\}$ the union of two quadric surfaces Q_1 & Q_2. $Q_1 \cap Q_2$ is an elliptic curve. We get:

$$\mathrm{Vol}_{sb}(\mathfrak{X}_{K(\infty)}) = [Q_1]_{sb} + [Q_2]_{sb} - [Q_2 \cap Q_1]_{sb}$$

$$= 2[\mathrm{Spec}\, k]_{sb} - [Q_2 \cap Q_1]_{sb}$$

this is $[\mathrm{Spec}\, k]$ iff $Q_2 \cap Q_1$ stably rational. Not possible

So a very general quartic surface is stably irrational.

A handy result: S Noetherian $\mathbb{Q}$-scheme. $X \to S$ smooth and proper. Then the set of $s \in S$ s.t

$X \times_S \bar{s}$ stably rational for $\bar{s}$ geometric point over $s \in S$

is a countable union of closed sets.

Example: $\{ft + q_1 q_2 = 0\} = X \subset \mathbb{A}^1 \times \mathbb{P}^3$. All fibers are smooth except $t=0$. So $X \setminus X_0 \longrightarrow \mathbb{A}^1 \setminus 0$

is smooth and proper. Above we saw that $\mathcal{X}_{K(\infty)}$ was stably irrational. This is a geometric fiber over the generic point so we have a non-stably rational fiber.

$\Longrightarrow$ Very general fiber is not stably rational.

$\Rightarrow$ This implies the existence of stably irrational fiber over the field k!

Can be applied to parameter spaces!

Example: $X_d \subset \mathbb{P}^n$ a smooth hypersurface of degree d.

If X_d is stably irrational then a very general hypersurface of degree d is stably irrational.

Take parameter space $\mathbb{P}^N$ for degree d hypersurfaces.

Let U be the smooth locus. Let $Y \to U$ be the universal family. Then $Y \to U$ is smooth, proper, and for some $x \in U$ the fiber is X_d which is st. irrational.

$\Rightarrow$ Very general fiber is stably irrational

$\Leftrightarrow$ very general hypersurface of degree d is stably irrational

WHAT HAVE WE DONE?

$\mathcal{C}$ a parameter space of vars$/k$. Want to show that very general members are stably irrational.

* Construct a strictly toroidal model $\mathcal{X}$ s.t

1) $\mathcal{X}_K$ smooth K-scheme of type $\mathcal{C}$

2) $\mathcal{X}_K$ satisfies $\displaystyle\sum_{E \in S(\mathcal{X})} (-1)^{\operatorname{codim} E} [E]_{sb} \neq [\operatorname{Spec} k]_{sb}$

Theorem $\Rightarrow$ $\mathcal{X}_K$ stably irrational over K.

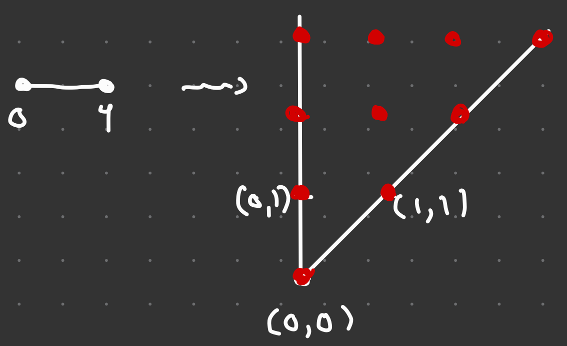
Corollary $\Rightarrow$ Get smooth stably irrational schemes in $\mathcal{C}$ over k.

Note: Using this one reduces the question of rationality to known examples. One cannot construct explicit examples.

Toric Degenerations

$\Delta \subset \mathbb{Z}^n$ lattice polytope of dim n. To any such we get a projective variety $\mathbb{P}_K(\Delta)$ with an ample line bundle $\mathcal{L}(\Delta)$.

Example

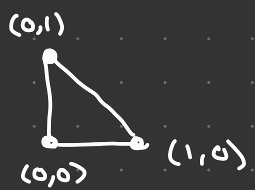


Lattice points in the cone defines a monoid M generated by $(0,1), (0,0), (1,0)$.

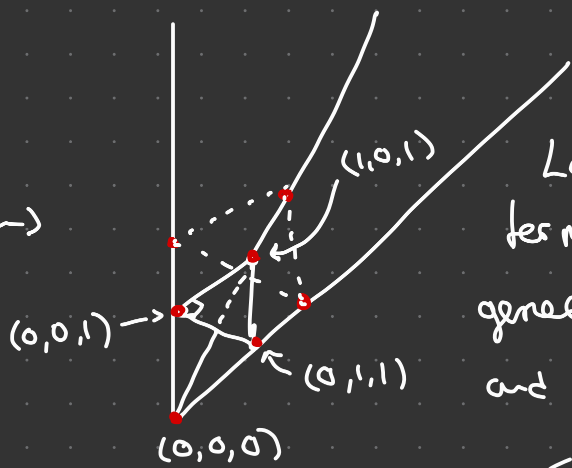
$K[M] = K[\chi^{(0,1)}, \chi^{(1,0)}]$ with multiplication $\chi^{(m_1,r_1)} \cdot \chi^{(m_2,r_2)} = \chi^{(m_1+m_2, r_1+r_2)}$ and $\deg \chi^{(m,r)} = r$.

$\mathbb{P}_k(\Delta) = \operatorname{Proj} k[M] \simeq \mathbb{P}^1$. $\quad \mathcal{L}(\Delta) = \mathcal{O}_{\mathbb{P}^1}(1)$

Example



$\rightsquigarrow$



Lattice points in this cone form a monoid M generated by $(0,0,1)$, $(0,1,1)$ and $(1,0,1)$,

$\rightsquigarrow$ Finitely generated k-algebra

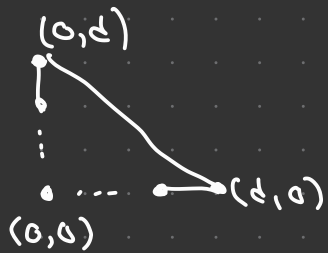
$$k[M] = k[\chi^{(0,0,1)}, \chi^{(0,1,1)}, \chi^{(1,0,1)}]$$

with multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ and $\deg \chi^{(s,t,r)} = r$.

$\mathbb{P}_k(\Delta) := \operatorname{Proj}(k[M]) \simeq \mathbb{P}^2$, $\mathcal{L}(\Delta) = \mathcal{O}_{\mathbb{P}^2}(1)$.

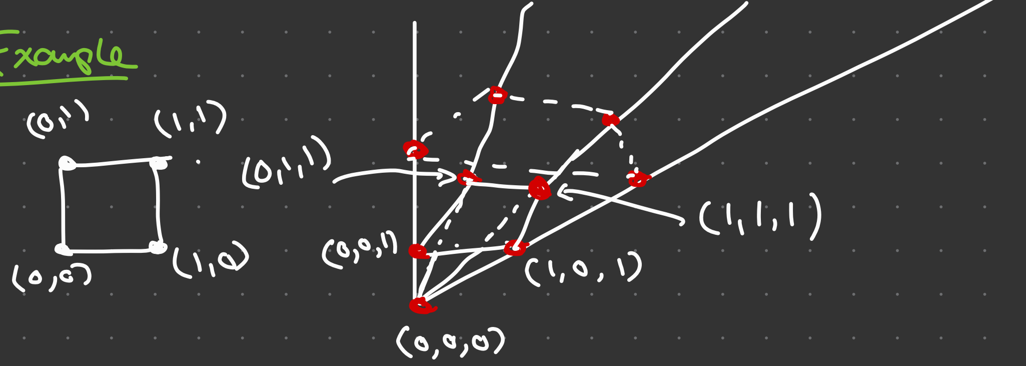
Example

Scaling to



gives $\mathbb{P}^2$ with $\mathcal{L}(\Delta) = \mathcal{O}(d)$.

Example



Monoid M generated by $(0,0,0)$ $(1,0,1)$ $(0,1,1)$ $(0,0,1)$ and $(1,1,1)$.

$$\rightsquigarrow \quad k[M] = k\left[\chi^{(0,0,1)}, \chi^{(1,0,1)}, \chi^{(0,1,1)}, \chi^{(1,1,1)}\right]$$

$$\simeq k[x,y,z,w] \big/ (yz - xw) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3.$$

$$\mathbb{P}_k(\Delta) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathcal{L}(\Delta) = \mathcal{O}(1,1).$$

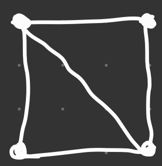
Degenerating the Toric Varieties

$\Delta \subset \mathbb{Z}^n$ lattic polytope. A Polyhedral subdivision of Δ is a set $\mathcal{P}$ of subpolytopes of Δ s.t

① $\alpha, \beta \in \mathcal{P} \Rightarrow \alpha \cap \beta \in \mathcal{P}$

- $\mathcal{P}$ is integral if $\alpha \in \mathcal{P}$ are all lattice polytopes.
- $\mathcal{P}$ is regular is there is a piecewise linear function $\Delta \to \mathbb{R}$ s.t the affine domains is the faces of $\mathcal{P}$.

Example



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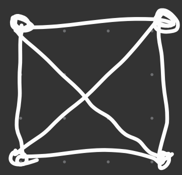

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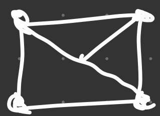
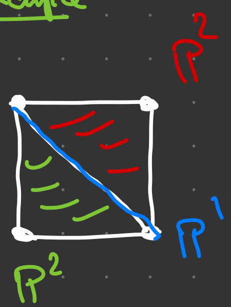


Not valid.

* Let X denote the zero locus of a very general section $s \in H^0(\mathcal{L}(\Delta))$.

Any integral regular polyhedral subdivision of the lattice polytope Δ induces a degeneration of X.

Example



X is a bidegree $(1,1)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \rightsquigarrow & \mathbb{P}^2 \cup \mathbb{P}^2 \text{ meeting in } \mathbb{P}^1 \\ \cup & & \cup \quad\;\; \cup \\ X & \rightsquigarrow & H_1 \cup H_2 \text{ meeting in a point.} \end{array}$$

* CCW TALK TODAY 5pm.