UiO : Department of Mathematics

University of Oslo

Around Ragsdale conjecture

and the topology of real plane algebraic curves

Cédric Le Texier

February 23, 2021

Positive and negative locus of a real polynomial

Let $C = V(f) \subset \mathbb{P}^2_{\mathbb{R}}$ be a smooth real algebraic curve of $\mathbb{P}^2_{\mathbb{R}}$ defined by a homogeneous polynomial $f \in \mathbb{R}[x_0, x_1, x_2]$ of even degree.

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- What is the repartition of these connected components ?

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The answer was already known up to degree 5, so the question was initially asked for degree 6 curves

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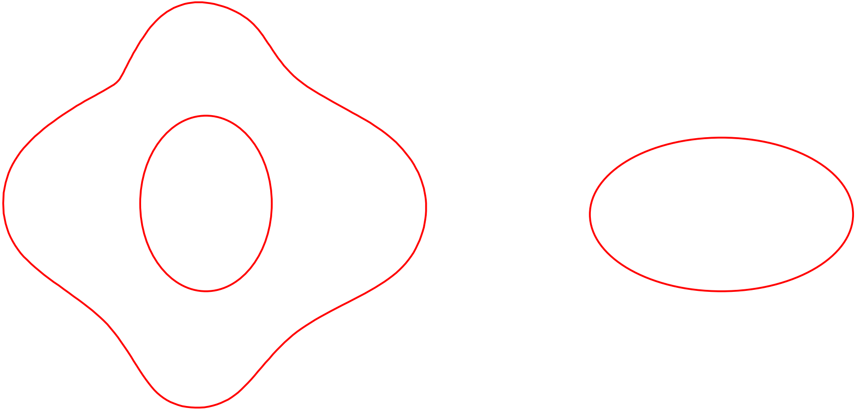
- In degree 5, the possibilities are $J, J \sqcup 1, J \sqcup 2, J \sqcup 1\langle 1\rangle, J \sqcup 3, J \sqcup 4, J \sqcup 5, J \sqcup 6$.

Bézout type restriction



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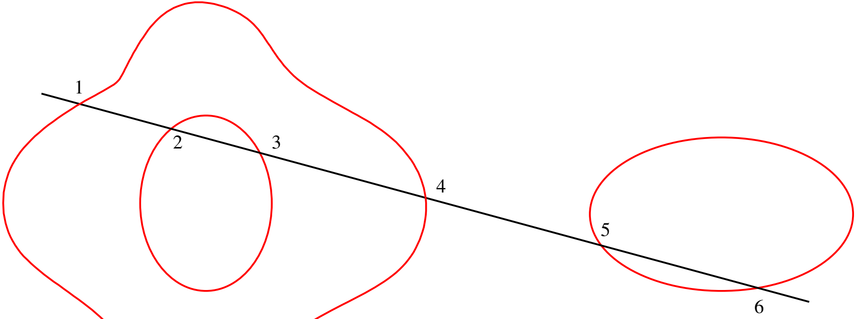
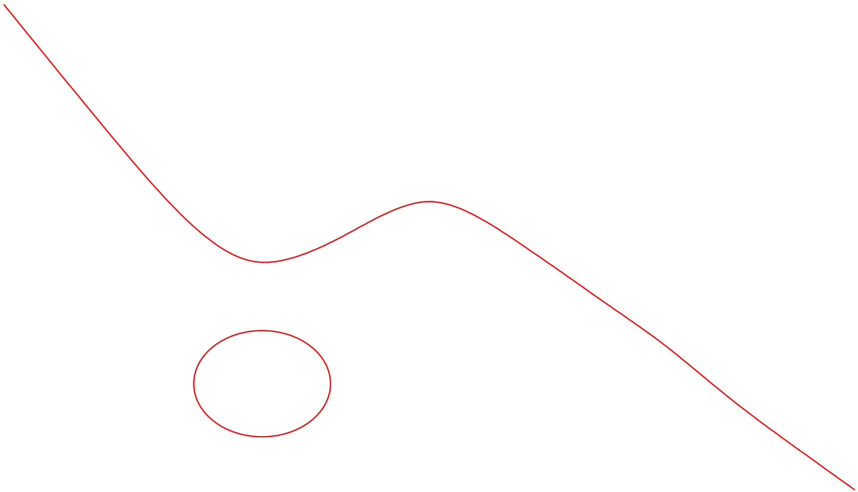


Harnack's construction



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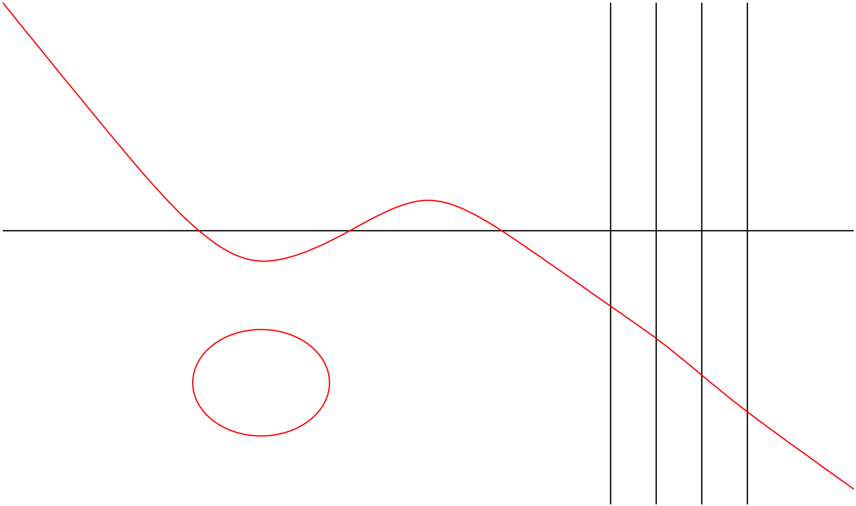

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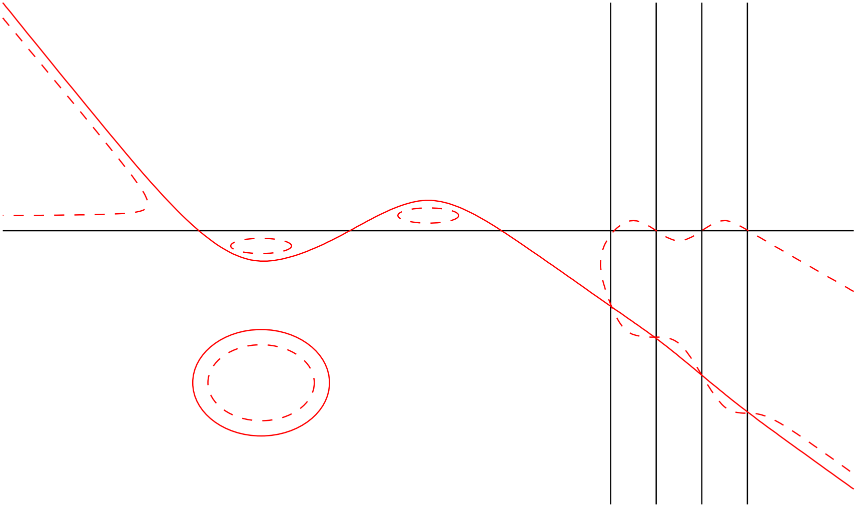
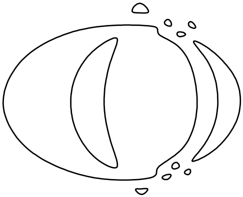
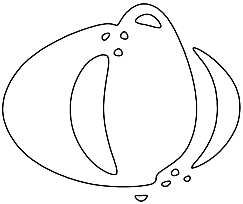


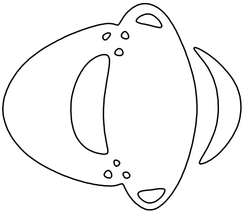
Examples in degree 6



Harnack's sextic



Gudkov's sextic



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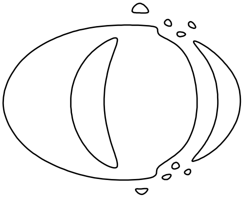
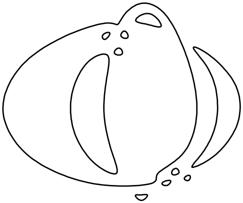


Harnack's sextic

$9 \sqcup 1\langle 1\rangle$



Gudkov's sextic

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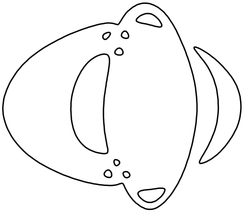

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Topological restrictions and Ragsdale conjecture

- Harnack's bound: For C be a real algebraic curve,

$$b_0(\mathbb{R}C) \leq g(\mathbb{C}C) + 1.$$

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$$p - n \leq \frac{3k^2 - 3k}{2} + 1, \quad n - p \leq \frac{3k^2 - 3k}{2}$$

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- Ragsdale/Petrovsky's conjecture: with the same hypotheses as above,

$$p \leq \frac{3k^2 - 3k}{2} + 1, \quad n \leq \frac{3k^2 - 3k}{2}(+1).$$

Return to degree 6 examples

The genus of a degree $2k = 6$ plane curve is 10, hence a maximal curve (in Harnack's sense) of degree 6 has 11 ovals.

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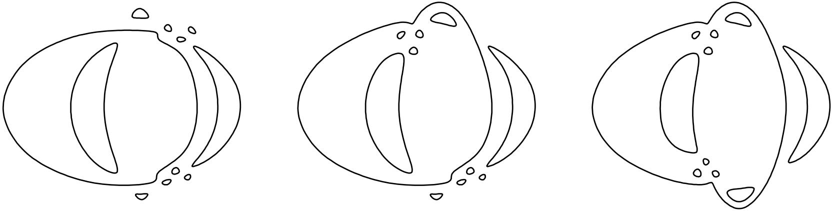

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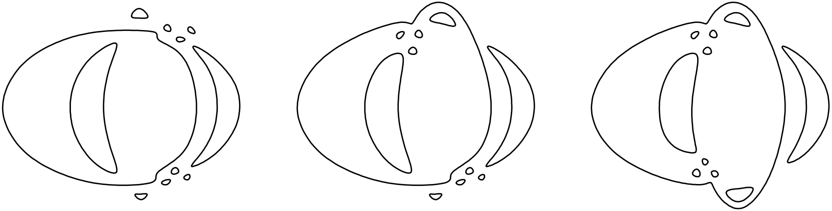

$p = \frac{3k^2-3k}{2} + 1 = 10$

(Not interesting here)

$n = \frac{3k^2-3k}{2} = 9$

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Starting from degree 8, no complete classification is known (at least in algebraic case).

First "small" counter-examples in degree 8

Theorem (Viro, 1980)

For every $k \geq 4$ even, there exist maximal curves of degree $2k \geq 8$ satisfying $n = \frac{3k^2-3k}{2} + 1$.

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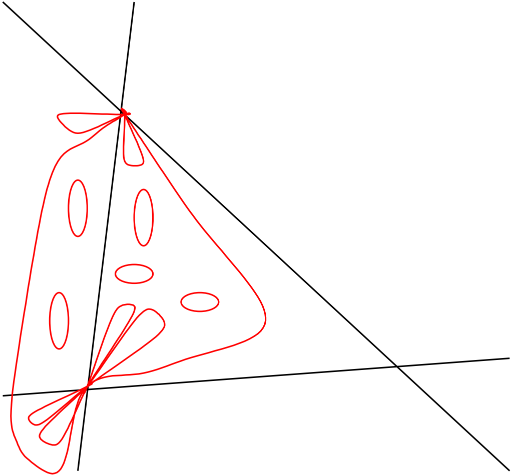
Theorem (Viro, 1980)

For every $k \geq 4$ even, there exist maximal curves of degree $2k \geq 8$ satisfying $n = \frac{3k^2-3k}{2} + 1$.

We get here that Ragsdale conjecture is false, but Petrovsky's conjecture is still satisfied.

Smoothing complicated singularities



Itenberg's counter-examples

Theorem (Itenberg, 1993)

For every $k \geq 5$, there exists a non-singular real algebraic curve of degree $2k$ satisfying

$$p = \frac{3k^2 - 3k}{2} + 1 + \left\lfloor \frac{(k-3)^2 + 4}{8} \right\rfloor .$$

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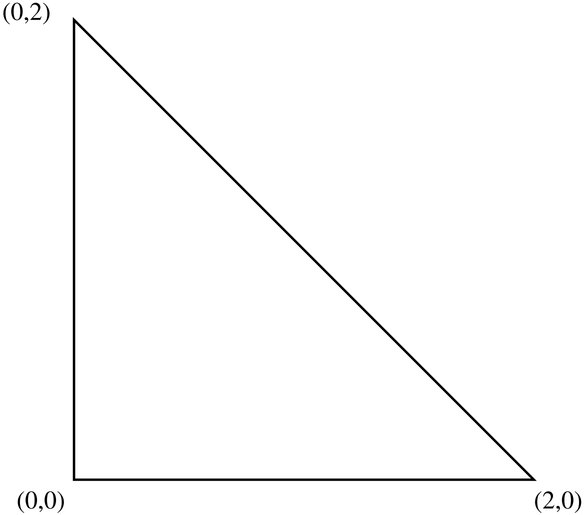
Combinatorial patchworking



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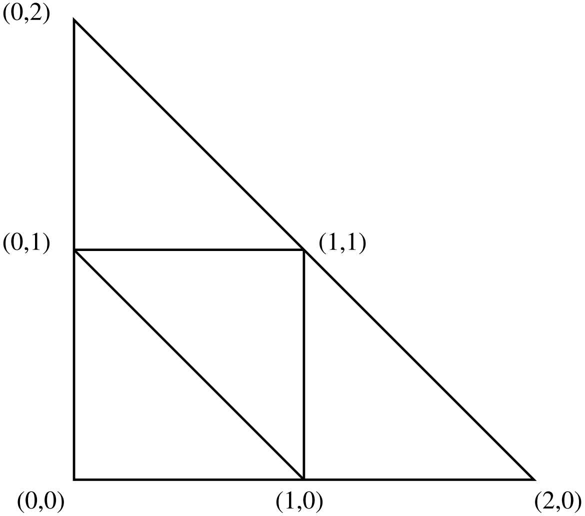

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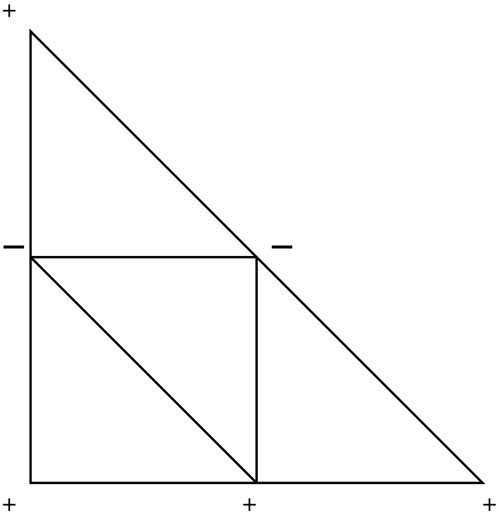

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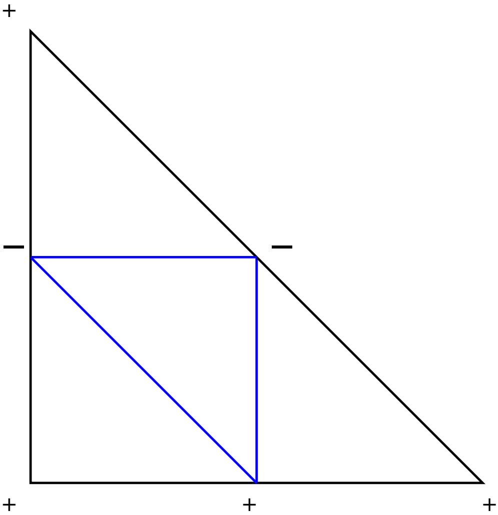

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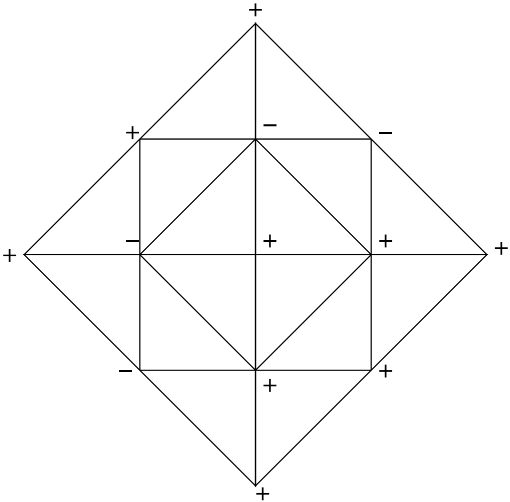

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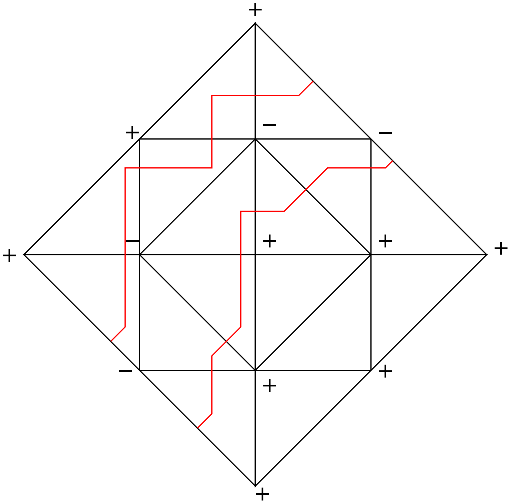

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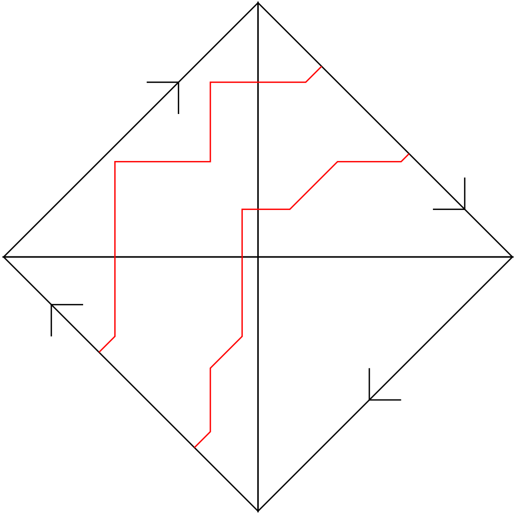
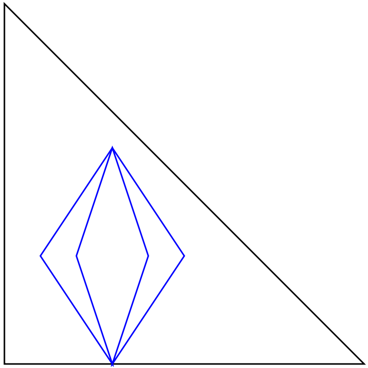
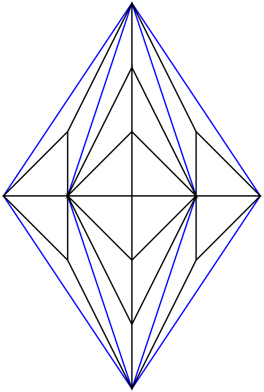


Itenberg's construction in degree 10





Open questions

Combining Harnack and Petrovsky's inequalities, we obtain the bounds

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Questions

- Is the Harnack-Petrovsky bound sharp ?
- Do we have counter-examples for maximal curves ?

Partial answers

Theorem (Brugallé, 2006)

The Harnack-Petrovsky bound is asymptotically sharp.

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Theorem (Haas, 1997)

Any maximal curve obtained by combinatorial patchworking satisfies

$$p \leq \frac{3k^2 - 3k}{2} + 1, \quad n \leq \frac{3k^2 - 3k}{2} + 4.$$

No example of maximal curve with $n > \frac{3k^2-3k}{2} + 1$ is known.

Best examples in low degree

Theorem (Haas, 1995)

For every $k \geq 5$, there exists a non-singular real algebraic curve of degree $2k$ satisfying

$$p = \frac{3k^2 - 3k}{2} + 1 + \left\lfloor \frac{k^2 - 7k + 16}{6} \right\rfloor .$$

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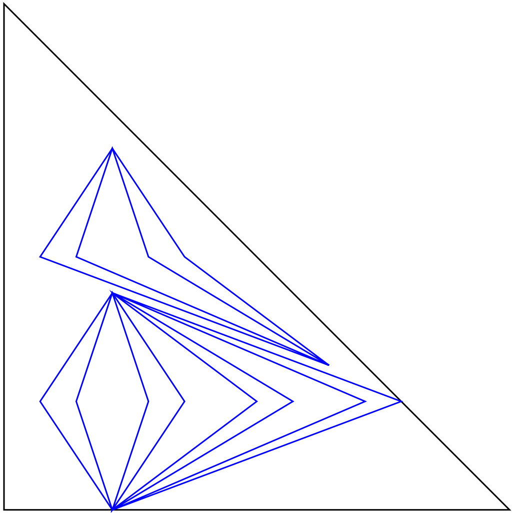
Construction in degree 14



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