# An Introduction to Motivic Homotopy theory

Thursday, 18 March 2021 09:58

## The origin of Motives (~1960)

$X \in Sm/_F$, Smooth (seperated) schemes of finite type over $F$.

(For simplicity char $F = 0$.)

- Betti cohomology: $F \overset{u}{\hookrightarrow} \mathbb{C}$ (Choose an embedding)

  $H^*_B(X) := H^*_{sing}(X(\mathbb{C}), \mathbb{Z})$ (abelian group) ((p,q) – with Hodge decomposition)

- (Algebraic) de Rham

  $H^*_{dR}(X) := \mathbb{H}_{zar}(X, \Omega^\bullet_{X/F})$ (F-vector space)

- l-adic cohomology $(F \hookrightarrow \bar{F})$

  $H^*_\ell(X) := H^*_{et}(X, \mathbb{Q}_\ell) = \varprojlim H^*_{et}(X, \mathbb{Z}/\ell\mathbb{Z}) \otimes \mathbb{Q}_\ell$

## Comparison maps

- $H^*_B(X) \otimes \mathbb{C} \simeq H^*_{dR}(X) \otimes \mathbb{C}$
- (Simililar phenomena happens in char $F \neq 0$)

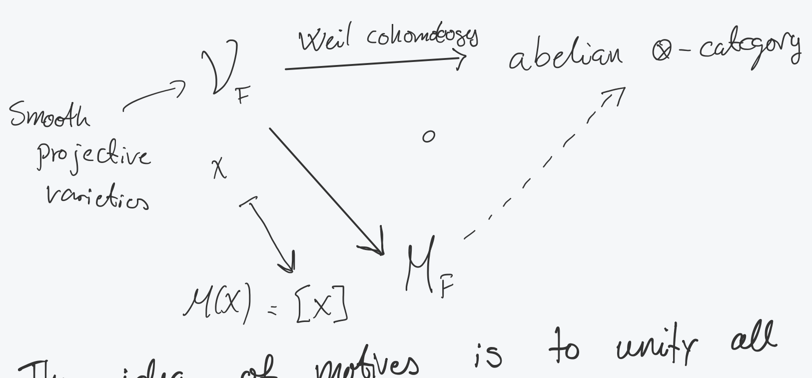
## Similarities

- Contravariant functors
- Take values in different F-vector spaces
- $\dim H^0_?(X) = \dim H^{\dim 2X}(X) = 1$
- Satisfy Poincaré duality, Künneth theorem
- ⋮

3 (Properties of Weil cohomology theories)

## Grothendiecks dream

There should exist a category of "pure"-Motives throug which all Weil cohomology theories should factor.




The idea of motives is to unify all known cohomology theories, and also to give precise meaning to

- [point]
- [projective line] = [line] + [poin]
- [projective plane] = [plane] + [line] + [point]

  $(\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0)$

**Thm (Grothendieck)**

The category of (pure)-motives exists iff the standard conjecture of algebraic cycles holds

RMK: When char $F = 0$, the standard conjectures are implied by the Hodge conjecture.

(No serious progress since 1960-1970!)

Deligne proposed to first construct the category of derived motives, that is the derived category of $M_F$.

Vocvodsky created a category of $DM$ that should function as the derived category of $M_F$.

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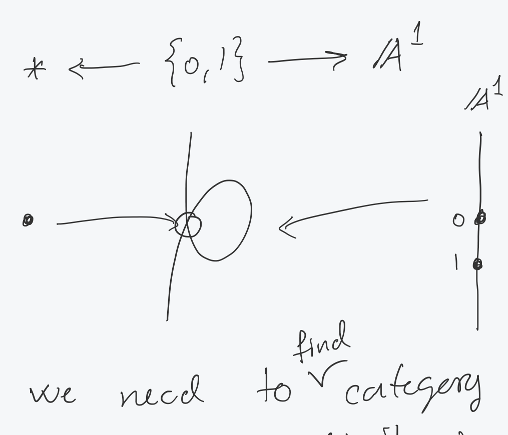
## Homotopy theory on Schemes.

It turns out that $Sm/_F$ is not good enough.

We do not have all small colimits:

"The non-existance of contractions"

$* \leftarrow \{0,1\} \rightarrow \mathbb{A}^1$




Therefore we need to find category with all small limits and colimits in which our category of schems embedds.

$$Sm/_F \hookrightarrow PreShv(Sm/_F)$$

$$X \longmapsto Hom_{Sm/_F}(-, X) =: R_X$$

This is given by the Yoneda embedding.

This has a disadvantage.

$X = U \cup V$ a Zariski open covering

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

X is the categorical union of U and V.

$$\begin{array}{ccc} R_{U\cap V} & \longrightarrow & R_V \\ \downarrow & & \downarrow \\ R_U & \longrightarrow & R_X \end{array}$$

But this is not a pushout diagram, that is $U \cup_{PreShv} V \longrightarrow X$ is not an isomorphism.

So therefore we introduce the Nisnevich topology.

Zariski ⊂ Nisnevich ⊂ étale.

$Shv_{Nis}(Sm/_F) \subset PreShv(Sm/_F)$.

Our next task is to find a suitable replacement for $I$.

Voevodsky chooses $\mathbb{A}^1$.

**Def** A presheaf is $(\mathbb{A}^1)$-homotopy invariant if

$$\mathcal{F}(X \times \mathbb{A}^1) \longrightarrow \mathcal{F}(X)$$

is an isomorphism.

Voevodsky inverts these (Model categories) (Quillen 1967') morphism, and with some technicalities, he constructs $DM$ as a derived category of

$$DM^{eff,-}_{Nis} \subset D^-(L_{\mathbb{A}^1} L_{Nis} PreShv(Sm/_F))$$

$DM^{eff,-}_{Nis}$ has many good properties, but sadly this is not the derived category of $M_F$.

**Thm (Voevodsky):** ([Proposition, 4.3.8, "Triangulated category of motives over a field"]

$DM^{eff}_{gm}$ has no reasonable t-structure.

Precise statement:

k field s.t. there exists a conic X over k with no k-rational point, then $DM^{eff}_{gm}$ has no reasonable t-structure.

$(D_{\geq 0} \cap D_{\leq 0} = D_0 = \mathcal{M})$)

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Choosing $\mathbb{A}^1$ as the replacement for $I$, restrict our theory to $\mathbb{A}^1$-invariant phenomena. There are a lot of cohomology theories that are not $\mathbb{A}^1$-invariant.

- P-adic cohomology

  $H^*_{et}(X, \mathbb{Z}_p)$, $1/p \notin \mathcal{O}_X$

- Hodge cohomology

  $H^n(X, \Omega^j)$.

However, they are $\mathbb{P}$-invariant.

Problem: $\mathbb{P}^1$ is not contractible.

We construct a gadget $\bar{\square} := (\mathbb{P}^1, \infty)$.

This does not live in algebraic geometry, but in log geometry.