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Tropical homology manifolds

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of professor Kris Shaw. This work was supported by the Trond Mohn Foundation project "Algebraic and Topological Cycles in Complex and Tropical Geometries".

The thesis is a collection of three papers, presented in chronological order of writing. The common theme to them is a study of certain topics within tropical cohomology theory. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work. The second paper is joint with Omid Amini, Matthieu Piquerez, and Kris Shaw. I am the sole author of the first and third papers.

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Edvard Aksnes
Oslo, February 2024

Abstract

English

In this thesis, we study *tropical homology manifolds*, a type of space with multiple distinguished properties within tropical geometry, and in particular tropical cohomology theory. We quickly review these notions, as a background to the contents of the thesis.

Tropical geometry is a relatively new field of mathematics, tightly bonding to algebraic geometry, combinatorics and multiple other fields. While some of the ideas in tropical geometry find roots back to the seventies, the main impetus to the field came after the turn of the millennium, with the work of Mikhalkin relating tropical curve counting and Gromov–Witten invariants.

In tropical geometry, the *varieties* of classical geometry, such as smooth curves and surfaces, are replaced with piecewise linear objects called *tropical varieties*. In particular, the latter are locally given as polyhedral *fans*, consisting of cones glued together along common boundaries. The operation of *tropicalization* transforms a classical object into a tropical one, through which some of the invariants are preserved, such as dimension and degree. For instance, the tropicalization of a complement of a hyperplane arrangement yields the *Bergman* fan of the matroid of the arrangement. More generally, any matroid has a corresponding Bergman fan, and such fans are the building blocks of *tropical manifolds*.

Tropical cohomology is an invariant of a tropical variety, introduced in a joint article by Itenberg, Katzarkov, Mikhalkin and Zharkov. They show that, for a family of smooth varieties, tropical cohomology recovers information about the mixed Hodge structures, and hence cohomology, of the varieties, provided that its tropicalization is a tropical manifold. This is dependent on the simpler property that, for complements of hyperplane arrangements, the tropical cohomology of the corresponding Bergman fan is isomorphic to the Orlik–Solomon algebra, which in turn is isomorphic to the cohomology. Another property of the tropical cohomology of Bergman fans is that they satisfy *tropical Poincaré duality*, an isomorphism property between tropical homology and cohomology, as shown by Jell, Rau, Shaw and Smacka.

Finally, we note that the complexifications of complements of real hyperplane arrangements satisfy a certain equality between the sum of their mod two Betti numbers, making these *maximal varieties*. Rau, Renaudineau and Shaw have introduced real phase structures on the Bergman fans of matroids, which through the use of a spectral sequence leads to the bounds for the aforementioned Betti numbers for real algebraic varieties with tropical manifolds as tropicalization.

This thesis approaches properties of tropical cohomology satisfied by fans of matroids, with the perspective of finding more general spaces satisfying the above-mentioned properties. The thesis consists of three articles.

In the first article of this thesis, we study tropical Poincaré duality. For any rational balanced polyhedral fan, there is a tropical fundamental class, which induces cap products between tropical cohomology and tropical Borel–Moore homology. When all these cap products are isomorphisms, the fan is said to be a *tropical Poincaré duality space*. If for each face, the corresponding star fan are also tropical Poincaré duality spaces, the fan is called a *local tropical Poincaré duality space* or a *tropical homology manifold*.

The article gives necessary conditions for fans to satisfy tropical Poincaré duality, as well as a classification in dimension one. Moreover, under a vanishing condition for Borel–Moore homology, we show that when all the stars of proper faces of a fan satisfy tropical Poincaré duality, so does the fan itself. Using this, we give necessary and sufficient conditions for a fan to be a tropical homology manifold, and thereafter construct abstract balanced polyhedral spaces satisfying tropical Poincaré duality using these fans.

In the second article of this thesis, joint with Amini, Piquerez and Shaw, we investigate under which conditions the tropical cohomology of the tropicalization of a variety computes its cohomology. Given the tropicalization of a complex subvariety of the torus, we define a morphism between the tropical cohomology and the cohomology of their respective tropical compactifications. Such a variety is *cohomologically tropical* if this map is an isomorphism for all closed strata of the tropical compactification.

We define *wunderschön* varieties as ones where, for a tropical compactification of the variety, the open strata are all connected, with pure mixed Hodge structures concentrated in the maximum possible weight. We show that a schön subvariety of the torus is cohomologically tropical if and only if it is wunderschön and its tropicalization is a tropical homology manifold. Moreover, we study other properties of cohomologically tropical and wunderschön varieties, showing that in a semistable degeneration to an arrangement of cohomologically tropical varieties, the Hodge numbers of the smooth fibers are computed by the tropical cohomology of the tropicalization, by extending the results of Itenberg, Katzarkov, Mikhalkin, and Zharkov.

In the third article of this thesis, we study arrangements of curves from a tropical perspective. We first define *arroids* as an abstract axiom set encoding the intersection properties of arrangements of curves, generalizing the definition of matroids of rank three which can come from line arrangements. Under the assumption that the curves in an arrangement intersect pairwise transversely, i.e. forbidding higher order intersections, we show that the tropicalization of the complement is determined by the associated arroid, by constructing abstractly *arroid fans*. Moreover, drawing upon the first and second papers of the thesis, we study which arroid fans are tropical homology manifolds, and give some conditions for when the complement of an arrangement of curves is cohomologically tropical. Finally, we give criteria for when the complement is a maximal variety in terms of tropical geometry, and using this, we construct a family of examples illustrating recent work of Ambrosi and Manzaroli.

It is a central recurring theme throughout the thesis that tropical homology

manifolds have rich geometry and combinatorics, mirroring some of the properties of Bergman fans of matroids. They therefore lend their name to the thesis title.

Norsk

I denne avhandlingen studerer vi *tropiske homologimangfoldigheter*, som er en type rom med flere særeskilte egenskaper innen tropisk geometri, og det er spesielt tropisk kohomologiteori som vektlegges. Som bakgrunn for avhandlingens innhold, begynner vi med en oppsummering av ovennevnte begrep.

Tropisk geometri er et relativt nytt fagfelt innen matematikk med sterk knytning til algebraisk geometri, kombinatorikk, og flere andre fagfelt. Noen av ideene i tropisk geometri har opphav i arbeider fra syttitallet, men hoveddrivkraften i fagfeltet kom rundt tusenårsskiftet med Mikhalkins arbeid som knyttet tropisk kurvenummerering til Gromov–Witten invarianter.

I tropisk geometri erstattes *varietetene* fra klassisk geometri, slik som glatte kurver og flater, med stykkevise lineære objekter kalt *tropiske varieteter*. Disse sistnevnte er lokalt gitt ved polyhedralske *vifter* som består av kjegler limt langs delte render. *Tropikalisering* går ut på transformere et klassisk objekt om til et tropisk. Gjennom denne prosessen bevares noen av det opprinnelige objektets invarianter, slik som dimensjon og grad. For eksempel, tropikaliseringen av komplementet til et hyperplanarrangement gir *Bergman-viften* til arrangementets matroide. Mer generelt gir enhver matroide opphav til en tilhørende Bergman-vifte, og slike vifter er byggestenene til *tropiske mangfoldigheter*.

Tropisk kohomologi er en av invariantene til en tropisk varietet, som ble innført i en felles artikkel av Itenberg, Katzarkov, Mikhalkin og Zharkov. Sammen viser de at for en familie av glatte varieteter, gir tropisk kohomologi informasjon om varietetenes blandede Hodge-struktur, og dermed deres kohomologi dersom tropikaliseringen er en tropisk mangfoldighet. Dette avhenger av egenskapen om at, for et komplement av et hyperplanarrangement, er den tropiske kohomologien til den tilhørende Bergman-viften isomorf til Orlik–Solomon algebraen, som igjen er isomorf til kohomologien. En annen egenskap til Bergman-vifter er at de tilfredsstiller *tropisk Poincaré-dualitet*, en form for isomorfi mellom tropisk homologi og kohomologi, som bevist av Jell, Rau, Shaw og Smacka.

Merk også at kompleksifiseringen av komplementer av reelle hyperplanarrangement tilfredsstiller en viss ulikhet mellom summen av deres modulo-to Bettitall, som gjør disse til *maksimale varieteter*. Rau, Renaudineau og Shaw har definert reelle fasestrukturer for Bergman-vifter til matroider, som ved bruk av spektralsekvenser gir grenser for de ovennevnte Bettitallene for reele algebraiske varieteter med tropiske mangfoldigheter som tropikalisering.

Denne avhandlingen studerer de tropisk kohomologiske egenskapene som er tilfredsstilt av matroiders Bergman-vifter, med et det siktemål å finne rom som tilfredsstiller noen av egenskapene nevnt over. Avhandlingen består av tre artikler.

I avhandlingens første artikkel studerer vi tropisk Poincaré-dualitet. For enhver balansert rasjonal polyhedralsk vifte finnes det en tropisk fundamentalk-

lasse, som induserer cap produkter mellom tropisk kohomologi og tropisk Borel–Moore homologi. Når alle cap produkter er isomorfier, kaller vi viften et *tropisk Poincaré-dualitetrom*. Hvis stjerneviften til enhver side av en vifte også er tropiske Poincaré-dualitetrom, kalles viften for et *lokalt tropisk Poincaré-dualitetrom* eller en *tropisk homologimangfoldighet*.

Artikkelen gir nødvendige betingelser for at vifter skal være tropiske Poincaré-dualitetsrom, og en klassifisering i dimensjon en. Under betingelse om at Borel–Moore homologi forsvinner utenfor øverst grad, viser vi at dersom all stjerneviftene til alle sider av en vifte tilfredsstiller tropisk Poincaré-dualitet, så gjør selve originalviften det også. Ved hjelp av dette gir vi tilstrekkelige og nødvendige betingelser for at en vifte skal være en tropisk homologimangfoldighet. Deretter bygger vi abstrakte balanserte polyhedralske rom som tilfredsstiller tropisk Poincaré-duality takket være disse viftene.

I avhandlingens andre artikkel, i samarbeid med Amini, Piquerez og Shaw, undersøker når tropisk kohomologi til en varietets tropikalisering beregner dens kohomologi. Gitt tropikaliseringen av en torus kompleks undervarietet, definerer vi en morfi mellom den tropiske kohomologien og kohomologien til tilhørende tropiske kompaktifiseringer. Vi kaller en varietet for *kohomologisk tropisk* dersom denne avbildningen er en isomorfi for alle lukkede strata av den tropiske kompaktifiseringen.

Vi definerer en *wunderschön* varietet til å være en hvor, for en tropisk kompaktifisering av varieteten, alle de åpne strataene er sammenhengende med rene blandede Hodge-strukturer konsentrert i den maksimale mulige vekten. Vi viser at en schön undervarietet av torusen er kohomologisk tropisk hvis og bare hvis den er wunderschön og dens tropikalisering er en tropisk homologimangfoldighet. Vi studerer også andre egenskaper ved kohomologisk tropiske og wunderschöne varieteter, slik som å vise at i en semistabil degenerering til en samling kohomologisk tropiske varieteter, kan Hodge tallene til de glatte fibrene beregnes ved den tropiske kohomologien til tropikalisering ved å utvide resultatene til Itenberg, Katzarkov, Mikhalkin og Zharkov.

I avhandlingens tredje artikkel studerer vi komplement av kurvearrangementer fra et tropisk perspektiv. Først definerer vi *arroider*, som et abstrakt sett av aksiomer som koder inn snittingsegenskapene til kurvearrangementer, ved å generalisere definisjonen av rang-tre matroider som kommer fra linjearrangement. Under antakelsen om at kurvene snitter parvis transversalt, dvs. ved å forby høyere ordens snitt, viser vi at tropikaliseringen av komplementet bestemmes av den tilhørende arroiden ved å bygge abstrakte *arroidevifter*. Med blikk på avhandlingens første og andre artikler, studerer vi hvilke arroidevifter som er tropiske homologimangfoldigheter, og gir betingelser for når komplementet til et kurvearrangement er kohomologisk tropisk. Til slutt gir vi kriterier for når et slikt komplement er en maksimal varietet ved hjelp av tropisk geometri, og basert på dette, bygger vi en familie av eksempler som illustrer nylig arbeid av Ambrosi og Manzaroli.

Et gjentakende tema i avhandlingen er at tropiske homologimangfoldigheter har rik geometri og kombinatorikk som gjenspeiler noen av egenskapene til matroiders Bergman-vifter. Avhandlingen er derfor oppkalt etter dem.

Contents

Chapter 1
Introduction

In this thesis, we investigate how certain properties of matroids, viewed through the lens of tropical geometry, may be generalized.

The notion of *matroids* was introduced independently by Whitney and Nakasawa [Whi35; Nak09a; Nak09b; Nak09c], and provides an abstraction for the concept of independence in mathematical objects, particularly in linear algebra and graph theory [Oxl11]. A surprising aspect of matroid theory is the wealth of different definitions yielding the same underlying object: a matroid. The equivalence of these sometimes dramatically different definitions is affectionately called the *cryptomorphic* definitions of a matroid. For our purposes only the axiomatization in terms of *flats* will be necessary.

Tropical geometry lies at the crossroads of combinatorics and algebraic geometry. Mikhalkin's paper [Mik05], enumerating both real and complex curves in toric surfaces, initiated a lot of interest for tropical geometry and *tropicalization* of varieties. There are at least two approaches to the tropicalization of varieties, the primary two being through using non-archimedean valuations [MS15], or through Maslov dequantization [IMS09]. In tropical geometry, the operations of addition and multiplication of real numbers are replaced by maximum and addition respectively, yielding the *tropical* or *max-plus* semi-ring. This replacement also leads to a new type of geometry, where polynomial "equations" give rise to piecewise linear objects. A recurring theme, as illustrated by Mikhalkin's paper, is that some classical problems can be transformed into tropical counterparts sometimes yielding equivalent yet simpler formulations.

The interest for matroids in tropical geometry can be traced back to Sturmfels' work on the tropicalization of linear spaces [Stu02, Chapter 9]. Sturmfels shows that the tropical variety of an ideal generated by homogeneous linear forms is given entirely by the matroid of the ideal, giving a combinatorial description of the tropicalization, the *Bergman fan* of the matroid. These are named after Bergman's pioneering work, forming a root of tropical geometry [Ber71]. For realizable matroids, [Stu02, Theorem 9.6] notes that the Bergman fan is in fact a polyhedral fan thanks to an explicit description of the logarithmic limit set given in [BG84]. For arbitrary matroids, Sturmfels describes a set called the *Bergman complex* [Stu02, Theorem 9.12]. Taking the central cone over a Bergman complex, one recovers the corresponding Bergman fan. Explicit fan structures on the Bergman fan in terms of flags of flats of the underlying matroid were first described by Ardila and Klivans in [AK06], where topological and combinatorial properties of these fans are studied in detail. In [Spe08], Speyer describes the more general tropicalization of linear spaces over non-archimedean fields, yielding polyhedral complexes rather than fans, and corresponding to the more general concept of valuated matroids. However, the vertices of these complexes correspond to Bergman fans of matroids. More recently, Fink has shown that these tropical linear spaces are exactly the tropical varieties of degree one [Fin13].

Bergman fans of matroids have also garnered interest from the perspective of defining abstract tropical spaces. Based on a suggestion of Mikhalkin, a *tropical manifold* is a space locally given by tropical linear spaces, as studied in [MZ14; BIMS15; JRS18; IKMZ19; JSS19], mirroring the construction of manifolds as spaces locally given by euclidean space. The motivation behind this definition lies in the multiple nice properties Bergman fans satisfy. Of particular interest in this thesis are the strong properties on their *tropical homology and cohomology groups*, introduced by [IKMZ19], which can be associated to any fan (see Section 1.1.3 for a detailed introduction).

It was shown in [JRS18; JSS19; GS23] that Bergman fans have a certain kind of isomorphism between their tropical homology and cohomology groups, called *tropical Poincaré duality*. This property has been incorporated into a full *Hodge theory*-equivalent for tropical manifolds by [AP20]. Moreover, if a matroid arises from a complex hyperplane arrangement, tropical cohomology of the corresponding Bergman fan computes the *Orlik–Solomon algebra* of the matroid [Sha11; Zha13], which is isomorphic to the singular cohomology of the complement of the arrangement. Moreover, this relation between tropical and singular cohomology yields results in real algebraic geometry [RS23], thanks to the *maximality* of complements of hyperplane arrangements.

This thesis consists of three papers. The first paper studies which tropical varieties satisfy tropical Poincaré duality in the same manner as Bergman fans. Similarly, the second paper studies which complex varieties have their cohomology computed by tropical cohomology of their tropicalization, mirroring the phenomenon for complements of complex hyperplane arrangements. The third paper introduces an axiomatization of arrangements of curves, gives families of examples of fans satisfying the properties of the two first papers, and finishes by deducing a result in real algebraic geometry.

One of the recurring themes is that tropical varieties satisfying a strengthened version of tropical Poincaré duality, namely the *tropical homology manifolds*, are particularly amenable to proving results, containing some part of the flexibility of matroids, and are key to a deeper understanding of tropical cohomology. They therefore lend their name to the thesis title.

1.1 Preliminaries

Before turning to a detailed description of the papers in this thesis, we describe some of the main objects and concepts making repeated appearances.

1.1.1 Fans and polyhedral complexes

Throughout this thesis, we work with fans and polyhedral complexes. We begin by describing fans. While these have their own interesting combinatorics, see e.g.

[Zie95], our interest in fans stems from their role in tropical and toric geometry. The tropicalization of a variety over a trivially valued field is supported on a rational polyhedral fan [MS15, Corollary 3.5.5], and fans guide the construction of toric varieties [Ful93]. More generally, tropicalizing a variety over a non-trivially valuated field generally yields a set whose support is a polyhedral complex [MS15, Theorem 3.3.5]. These are locally supported on fans. Moreover, polyhedral complexes are the building blocks of abstract tropical spaces [Mik06].

Let N be a rank n free abelian group, the *lattice*, and let $M \coloneqq \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. For any (commutative and unital) ring R, let $N_R \coloneqq N \otimes_{\mathbb{Z}} R$ and M_R be the corresponding free R-modules. A *cone* σ in $N_{\mathbb{R}}$ is the set $\mathrm{cone}(v_1, \ldots, v_k) \coloneqq \{a_1 v_1 + \cdots + a_k v_k \mid a_i \geq 0\}$ of positive linear combinations of vectors v_i from the lattice N, such that σ does not contain a full linear subspace of $N_{\mathbb{R}}$. A cone τ is said to be a *face* of σ if there is an element $m \in N_{\mathbb{R}}$ such that $m(x) \geq 0$ for all $x \in \sigma$ and $\tau \coloneqq \{x \in \sigma \mid m(x) = 0\}$. A *fan* is a finite collection of cones, such that any two cones meet along a common face. In particular, all cones meet in the central minimal vertex.

A *rational half-space* H is a set of the form $\{x \in N_{\mathbb{R}} \mid \alpha \cdot x \geq \beta\}$, with $\alpha \in M$ and $\beta \in N$. A *polyhedron* σ is the intersection of finitely many rational half-spaces, and a *face* of a polyhedron σ is another polyhedron given as $\sigma \cap H$ for some half-space H containing σ. Finally, a *polyhedral complex* is a finite collection of polyhedra such that the intersection of any pair of polyhedra is either empty or a face of both. The *dimension* of a polyhedral complex Δ is the maximum of the dimensions of its polyhedra, and we denote by Δ_k the set of k-dimensional polyhedra of Δ. More generally, an abstract *rational polyhedral space* is a paracompact second countable Hausdorff topological space admitting charts so that it is locally a rational polyhedral complex, with extended affine $\mathbb{Z}$-linear transition maps.

1.1.2 Matroids

A common thread in this thesis is to consider the tropical geometry of fans of *matroids* and see in which wider context they apply. In our setting, the most convenient definition is in terms of *flats*.

Definition 1.1.1. A *matroid* M on a finite set E consists of a set $\mathcal{F}$ of subsets of E such that

- $\emptyset, E \in \mathcal{F}$,
- For any two $F_1, F_2 \in \mathcal{F}$, the intersection $F_1 \cap F_2$ is contained in $\mathcal{F}$,
- For $F \in \mathcal{F}$, the minimal-by-inclusion elements $F_1, \ldots, F_k \in \mathcal{F}$ containing F are such that the sets $F_1 \setminus F, \ldots, F_k \setminus F$ form a partition of $E \setminus F$.

The set E is called the *ground set*, typically labelled so that $E = \{1, \ldots, n\}$, and the elements of $\mathcal{F}$ are called *flats*. There are many sources of examples for matroids, ranging from graphs to linear algebra [Oxl11]. One particular class of examples, which will make multiple appearances later, is that of the matroids coming from *arrangements of hyperplanes.* One considers a base set E consisting of hyperplanes in some projective space, and the lattice of intersections of these hyperplanes forms the collection of flats of a matroid, see e.g. [OT92] for details.

The combinatorial properties of matroids have been a subject of study since their inception, however a recent surge of activity can be attributed to ideas of Huh, linking matroid theory to Hodge theory. In [Huh12], Huh proves that the coefficients of the chromatic polynomial of a graph form a log-concave, hence unimodal, sequence, answering a conjecture of Read. In [HK12], this was taken further, proving log-concavity of the characteristic polynomial coefficients for any representable matroid, making progress on a conjecture of Rota–Heron–Welsh. This generalization is based on intersection theory for the toric variety associated to the Bergman fan of the matroid. Finally, in [AHK18], the full Rota-Heron-Welsh conjecture is established: for any matroid, the coefficients of the characteristic polynomial form a log-concave sequence. This was achieved by defining the Chow ring of an arbitrary matroid, and proving that this ring satisfies a version of Poincaré duality, the Hodge-Riemann bilinear relations, and the hard Lefschetz theorem. It is suggested in [Huh18] that such “Chow rings”, satisfying the above three properties should be the underlying reason for the log-concavity of many sequences which arise in mathematics.

In this thesis, the main objects based on matroids which we use are certain rational polyhedral fans, the *Bergman fans*, constructed as follows. Using the ground set, one may form a lattice $N \coloneqq \mathbb{Z}^{|E|}$ with basis $e_1, \dots, e_n$ indexed by the elements of E. Each subset S of the ground set E gives a vector e_S in the vector space $N_{\mathbb{R}} \cong \mathbb{R}^n$, taking the form

$$e_S \coloneqq \sum_{i \in S} e_i.$$

Moreover, a *flag* of flats is a collection $F_\bullet$ of flats of the form $F_\bullet = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_l = E\}$ for some l. To a flag $F_\bullet$, one associates the cone

$$\sigma_{F_\bullet} = \operatorname{cone}(e_{F_1}, \dots, e_{F_l}).$$

Considering all flags of flats, this collection of cones forms a fan.

Definition 1.1.2 ([AK06])**.** For M a matroid, the set of cones $\sigma_{F_\bullet}$, for each flag of flats $F_\bullet$ of M, forms a fan, called the *Bergman fan* of the matroid.

More generally, one may consider abstract rational polyhedral spaces which are locally isomorphic to Bergman fans of matroids. Such spaces are called *tropical manifolds.*

1.1.3 Tropical (co)homology

We begin by describing the origins of tropical cohomology, before turning to the main definitions.

In [IKMZ19], Itenberg, Katzarkov, Mikhalkin and Zharkov introduce *tropical (co)homology.* Their main theorem relates tropical cohomology and singular cohomology in the following setting. Let $\pi\colon \mathfrak{X} \to \mathbf{D}^*$ be a family of subvarieties of $\mathbb{CP}^n$ parametrized over the disk $\mathbf{D}^*$. The authors adapt the construction of a semi-stable model from [KKMS73], giving a completed family $\pi\colon \overline{\mathfrak{X}} \to \mathbf{D}$ such that the fiber $\mathfrak{X}_0 := \pi^{-1}(0)$ is a reduced simple normal crossing divisor.

Each of the fibers $\mathfrak{X}_t = \pi^{-1}(t)$ of this family is a projective Kähler manifold, with cohomology groups admitting Hodge structures, and there is a precise sense in which these Hodge structures vary smoothly as a function of t, see e.g. [Voi02, Chapter 10]. However, the central fiber $\mathfrak{X}_0$ has a mixed Hodge Structure, which does not directly correspond to a variation in the family. Schmid [Sch73] and Steenbrink [Ste76; Ste77], using different methods, show the existence of a *limit mixed Hodge structure*, compatible with both the Hodge structures of the smooth fibers and the mixed Hodge structure of the central fiber. In Steenbrink's approach, there is a space $\mathfrak{X}_\infty$, the *canonical fiber*, with a mixed Hodge structure on its cohomology, the limit mixed Hodge structure. The weight filtration on $H^\bullet(\mathfrak{X}_\infty)$ induces a spectral sequence, degenerating on the second page, where the E_1-page is given in terms of the cohomology of the components of the central fiber $\mathfrak{X}_0$.

In [IKMZ19], the authors show that when the tropicalization of the family is a tropical manifold, the E_1-page of this spectral sequence is entirely computable in terms of the tropical cochain complexes of their tropicalization. The limit mixed Hodge structure spectral sequence degenerates at the E_2-page, giving the following connection between tropical cohomology and the limit mixed Hodge structure.

Theorem 1.1.3 ([IKMZ19, Theorem 1])**.** *Let $\pi\colon \mathfrak{X} \to \mathbf{D}$ be as above. Assume the tropicalization X of $\mathfrak{X}$ is a tropical manifold. Then the tropical cohomology groups $H^{p,q}(X)$ are isomorphic to the associated graded groups $\mathrm{Gr}^W_{2p} H^{p+q}(\mathfrak{X}_\infty)$ of the weight filtration on limit mixed Hodge structure of the canonical fiber $\mathfrak{X}_\infty$.*

It also follows from the assumptions of the theorem that the Hodge numbers $h^{p,q}(\mathfrak{X}_t)$ of the generic fibers of the family are equal to the dimensions of the tropical cohomology groups $H^{p,q}(X)$ [IKMZ19, Corollary 1].

One of the key ingredients of the proof of Theorem 1.1.3 is a result of [Zha13] which compares the cohomology of the complement of an arrangement of hyperplanes, and the tropical cohomology of the tropicalization of this complement. In the study of hyperplane arrangements, it is a theorem of Orlik and Solomon [OS80] that the cohomology of the complement of a hyperplane arrangement is determined entirely by the matroid of the arrangement, giving rise to the Orlik–Solomon algebra. Zharkov shows that tropical cohomology of the fan of a matroid computes the Orlik–Solomon algebra. Another more combinatorial proof, based on iterating tropical modifications, was given by Shaw [Sha11].

We now turn to defining tropical (co)homology, which has been studied extensively in recent years, also beyond the origins described above [Zha13; MZ14; JRS18; IKMZ19; JSS19; AP20; AP21; ARS21; Mik21; Yam21; Aks23; GS23; Mik23; RS23]. In the following, we will describe the cellular approach to the restricted case of a polyhedral complex, as described in Section 1.1.1, which is most convenient for concrete computations.

Let Δ be a polyhedral complex in $N_{\mathbb{R}}$, for some lattice N. For each polyhedron σ of Δ, let $L(\sigma)$ be the saturated sublattice parallel to σ. For each $p \geq 1$, the *p-th multi-tangent space* $\mathcal{F}_p(\sigma)$ is the group defined by

$$\mathcal{F}_p(\sigma) \coloneqq \sum_{\sigma \preccurlyeq \gamma} \bigwedge^p L(\gamma) \subseteq \bigwedge^p N,$$

where the subspace sum is taken over all polyhedra γ of Δ containing σ, which we denote using the notation $\sigma \preccurlyeq \gamma$. For τ a face of σ, subspace inclusion induces a map $\iota_{\sigma \succcurlyeq \tau} \colon \mathcal{F}_p(\sigma) \hookrightarrow \mathcal{F}_p(\tau)$. Dualizing these groups and maps, one obtains the *p-th multi-cotangent spaces* $\mathcal{F}^p(\sigma)$, with maps $\rho_{\tau \preccurlyeq \sigma} \colon \mathcal{F}^p(\tau) \to \mathcal{F}^p(\sigma)$. Moreover, for any commutative ring R, one may tensor to get R-modules $\mathcal{F}_p^R(\sigma)$, $\mathcal{F}_R^p(\sigma)$ and R-module maps.

Assign an orientation to each cone σ, and for each face τ of σ, let $\operatorname{sign}(\tau, \sigma)$ be 1 if the chosen orientations of τ and σ are compatible, and -1 otherwise. Now for a give commutative ring R and each $p \geq 1$, the *p-th tropical Borel–Moore complex* $C_{p,\bullet}^{BM}(\Delta; R)$ is the complex

$$0 \longrightarrow \oplus_{\alpha \in \Delta_d} \mathcal{F}_p^R(\alpha) \xrightarrow{\partial_d} \oplus_{\beta \in \Delta_{d-1}} \mathcal{F}_p^R(\beta) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} \bigoplus_{v \in \Delta_0} \mathcal{F}_p^R(v) \longrightarrow 0.$$

The differential ∂_k is defined as the sum of its components $(\partial_k)_{\gamma,\delta} \colon \mathcal{F}_p^R(\gamma) \to \mathcal{F}_p^R(\delta)$, which is given by $\operatorname{sign}(\delta, \gamma) \cdot \iota_{\delta \prec \gamma}$ if δ is a face of γ, and 0 otherwise. The *p-th compact support cochain complex* $C_c^{p,\bullet}(\Delta; R)$ is defined by dualizing all maps and groups of the above complex.

The *p-th tropical chain complex* $C_{p,\bullet}(\Delta; R)$ is defined similarly to the Borel–Moore, with the restriction that only compact polyhedra are considered, i.e. $C_{p,q}(\Delta; R) \coloneqq \oplus_{\sigma \in \Delta_q^c} \mathcal{F}_p^R(\sigma)$, where Δ_q^c is the set of compact q-dimensional polyhedra of Δ. The differentials are defined as for $C_{p,\bullet}^{BM}(\Delta; R)$, and the corresponding dual construction $C^{p,\bullet}(\Delta; R)$ is the *p-th tropical cochain complex*.

The *tropical Borel–Moore homology* group $H_{p,q}^{BM}(\Delta; R)$ is the q-th homology group of the complex $C_{p,\bullet}^{BM}(\Delta; R)$, while the *tropical compact support cohomology* group $H_c^{p,q}(\Delta; R)$ is the q-th cohomology group of the complex $C_c^{p,\bullet}(\Delta; R)$.

The *tropical homology* group $H_{p,q}(\Delta; R)$ is the q-th homology group of the complex $C_{p,\bullet}(\Delta; R)$, and the *tropical cohomology* group $H^{p,q}(\Delta; R)$ is the q-th cohomology group of the complex $C^{p,\bullet}(\Delta; R)$. In particular, it follows that for a fan Σ and any $q > 0$, all higher tropical homology and cohomology modules, $H_{p,q}(\Sigma; R)$ and $H^{p,q}(\Sigma; R)$ respectively, are trivial. We denote by $H^k(\Sigma; R)$ the direct sums of the form $\oplus_{p+q=k} H^{p,q}(\Sigma; R)$ over all pairs p, q, and likewise for the other (co)homology modules.

The above homology and cohomology groups are defined for arbitrary polyhedral complexes, however they carry additional significance for complexes which are *tropical*. A polyhedral complex is *weighted* if it is equipped with an integer weight for each of its top-dimensional faces. Moreover, a weighted polyhedral complex is *tropical* if it is *balanced* in the sense of [BIMS15, Definition 5.7] and [MS15, Definition 3.3.1]. Fans satisfying this condition are more commonly known as *Minkowski weights* [FS97].

We note also that there are similarities between tropical (co)homology as introduced here, to work in the context of the Gross-Siebert program [Gro11]. Such parallels and potential equivalences have been explored by Ruddat [Rud21] and Yamamoto [Yam21].

1.1.4 Tropical Poincaré duality

From the perspective of tropical homology, for each set of weights w making a d-dimensional polyhedral complex Δ balanced, there is a corresponding *fundamental class* $[\Delta, w]$ in the top integral tropical Borel–Moore homology group $H^{BM}_{d,d}(\Delta;\mathbb{Z})$. In fact, this leads to an equivalent formulation of the balancing condition in terms of the existence of fundamental classes, see [MZ14, Proposition 4.3] and [JRS18, Remark 4.9]. From this point of view, the condition that a weighted polyhedral complex is tropical is akin to the orientability of manifolds. Beyond the case of $\mathbb{Z}$-coefficients, the same reasoning applies to more general commutative unital rings, as first studied in [JRS18] and expanded upon in the first paper of this thesis.

As in the case of topological manifolds, the fundamental class $[\Delta, w]$ can be used to relate tropical cohomology and tropical Borel–Moore homology. One may define a *cap product*

$$\frown : H^{p,q}(\Delta;\mathbb{Z}) \to H^{BM}_{d-p,d-q}(\Delta;\mathbb{Z}),$$

for each pair $0 \leq p, q \leq d \coloneqq \dim \Delta$. When all these maps are isomorphisms, the tropical variety is said to satisfy *tropical Poincaré duality*, and such a tropical variety is called a *tropical Poincaré duality space*. This duality was first introduced for $\mathbb{R}$-coefficient tropical (co)homology in [JSS19], which shows that tropical manifolds satisfy tropical Poincaré duality with $\mathbb{R}$-coefficients. This is achieved by showing that Bergman fans of matroids satisfy tropical Poincaré duality, together with a Mayer–Vietoris-type argument. Moreover, [JRS18] shows that Bergman fans and tropical manifolds in fact satisfy tropical Poincaré duality with $\mathbb{Z}$-coefficients, as described above. Note that, for a fan Σ, the triviality of the higher cohomology groups implies that the only non-trivial morphisms are of the form $\frown : H^{p,0}(\Delta;\mathbb{Z}) \to H^{BM}_{d-p,d}(\Sigma;\mathbb{Z})$.

Furthermore, we note that the recent work [AP20] of Amini and Piquerez establishes a full "Kähler package" for smooth projective tropical cycles, and relates tropical Poincaré duality of the canonical compactifications of Bergman fans of matroids to the Poincaré duality of the Chow ring of a matroid from [AHK18].

1.1.5 Three properties of matroids

With the above preliminaries in mind, we are now in a position to give a brief overview of the following three properties of matroids. Investigating extensions of these three properties is one of the goals of this thesis; this is done in three different papers:

1. In their work introducing tropical Poincaré duality, Jell, Shaw and Smacka show that this duality holds for tropical manifolds, and in particular Bergman fans of matroids [JSS19]. The first paper of this thesis investigates which more general spaces satisfy tropical Poincaré duality, giving some conditions for fans and constructions for abstract polyhedral spaces.

2. As was noted above, Zharkov [Zha13] and Shaw [Sha11] show that tropical cohomology of the fan of a matroid computes the Orlik–Solomon algebra, which, if the matroid is that of a hyperplane arrangement, in turn computes the cohomology of the complement of this arrangement [OS80]. The second paper of this thesis, joint with Amini, Piquerez and Shaw, investigates under which conditions similar isomorphisms between tropical cohomology of a tropicalization and singular cohomology of the original variety occur.

3. Complements of hyperplane arrangements are *maximal* in the sense of the Smith-Thom inequality relating Betti numbers of the real and complex parts, by work of Zaslavsky [Zas75]. In addition to studying complements of curve arrangements from a tropical perspective, the third paper of this thesis studies maximality conditions for such arrangements.

1.2 Paper one: Tropical Poincaré duality spaces

The central question of the first paper of this thesis, Paper I, is the following:

Which fans have tropical Poincaré duality?

We study this question from the perspective of an arbitrary (commutative and unital) coefficient ring R for tropical (co)homology, and first show that, when working with fans, the cap product is necessarily injective, see Proposition I.3.23. Moreover, for a zero-dimensional fan, i.e. a single point, tropical Poincaré duality is trivially satisfied.

In the case of one-dimensional fans, i.e. for which there is a single central vertex and a collection of emanating half rays, we achieve the following complete classification.

Theorem 1.2.1 (Theorem I.4.8)**.** *Let R be a commutative ring, and (Σ, w) an R-balanced fan of dimension one. Then (Σ, w) satisfies tropical Poincaré duality over R if and only if it is uniquely R-balanced and all the weights are units in R.*

Similarly, in the case of a codimension one tropical fan cycle, we show that its corresponding Newton polytope must be a simplex in Proposition I.4.11.

Going beyond the case of dimensions zero and one, one may consider fans satisfying tropical Poincaré duality for each *star* of one of its cones. For a fan Σ and cone γ, the *star* $\gamma_{\succeq}$ is the fan consisting of the whole linear subspace $L_{\mathbb{Z}}(\gamma)$ parallel to gamma, with polyhedral cells $\delta_{\pm} := \delta + L_{\mathbb{Z}}(\gamma)$ as a subspace sum, for each cone δ containing γ. To obtain a fan, one must subdivide these polyhedral cells so that there is a single central vertex and all cones are strictly convex. Note that the star of the central vertex of a fan recovers the fan itself.

A fan for which each star is a tropical Poincaré duality space will be called a *tropical homology manifold*. This is inspired by the distinction between *homology manifolds* and *Poincaré duality spaces* in topology, where the former property implies the latter, see e.g. [Ran11]. Any star of the Bergman fan for a matroid M is itself the Bergman fan of a matroid minor of M, and so the results of [JRS18; JSS19] imply that tropical manifolds are also tropical homology manifolds. Note that in the paper, this property is described as *Local tropical Poincaré duality*, and it is studied extensively in the case $R = \mathbb{Z}$ of integer coefficients in [AP21], where fans satisfying this property are called *tropically smooth*.

With the related aims of understanding tropical homology manifolds and studying the relation between tropical Poincaré duality for stars of a fan and tropical Poincaré duality for the fan itself, we show the following local-to-global type theorem.

Theorem 1.2.2 (Theorem I.5.4)**.** *Let R be a principal ideal domain, and (Σ, w) be an R-balanced fan of dimension $d \geq 2$, with $H_q^{BM}(\Sigma, \mathcal{F}_p^R) = 0$ for $q \neq d$, for all p. If $(\gamma_{\succeq}, w)$ satisfies TPD over R, for each $\gamma \in \Sigma$ with $\gamma_{\succeq} \neq \Sigma$, then (Σ, w) satisfies TPD over R.*

The above theorem is then applied inductively to all stars of the fan, which gives the following characterization of tropical homology manifolds.

Theorem 1.2.3 (Theorem I.5.10)**.** *Let R be a principal ideal domain, and (Σ, w) a d-dimensional R-balanced fan. Then Σ is a local TPD space over R if and only if $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for all $\gamma \in \Sigma$ and $q \neq d$, and for all faces β of codimension 1, the star fans $\beta_{\succeq}$ are TPD spaces over R.*

In the final part of the first paper, the above theorems are applied in the setting of *abstract tropical R-cycles* (see Definition I.6.1). These can be equipped with tropical homology and cohomology groups, and a balancing condition for abstract tropical R-cycles leads to cap products. Tropical manifolds are shown to satisfy TPD over $\mathbb{R}$ and $\mathbb{Z}$ in [JRS18; JSS19; GS23]. We say that an abstract tropical cycle is a *tropical homology manifold over R* if it is built from fans which are tropical homology manifolds over R, and using the Mayer–Vietoris arguments from [JRS18], we show that abstract tropical cycles which are tropical homology manifolds over R satisfy tropical Poincaré duality in Theorem I.6.5.

The salient difficulty in deeper understanding of tropical Poincaré duality and tropical homology manifolds lies in the vanishing condition on Borel–Moore homology, used in both Theorem I.5.4 and Theorem I.5.10, and in particular in giving geometric conditions on the fan to guarantee such vanishing. This difficulty is precisely formulated in I.5.13.

The second challenge, formulated in I.5.14, lies in understanding the difference between tropical Poincaré duality spaces and tropical homology manifolds. There are as of yet no examples of fans satisfying tropical Poincaré duality which are not also tropical homology manifolds. An argument based on the spectral sequence used in the proof of Proposition I.5.3 shows that such a fan would have to be at least three-dimensional.

1.3 Paper two: Comparing tropical and singular cohomology

The second paper of this thesis, Paper II, is a joint work with Omid Amini, Matthieu Piquerez, and Kris Shaw. One point of departure for this paper is the following question.

When is the cohomology of a variety isomorphic to the tropical cohomology of its tropicalization?

For example, as discussed in Section 1.1.3, Zharkov [Zha13] and Shaw [Sha11] have shown that, for the matroid of a hyperplane arrangement, one may use the tropical cohomology of the Bergman fan of the matroid to compute the matroid's Orlik–Solomon algebra. This latter algebra is isomorphic to the cohomology of the complement of the arrangement [OS80].

We approach the question in the following context. Let $\mathbf{X}$ be a complex subvariety of an algebraic torus $\mathbf{T}$ with lattice N, and $X \coloneqq \mathrm{trop}(\mathbf{X})$ its tropicalization. For the 0-th multi-cotangent sheaf $\mathcal{F}^0$, which is merely the constant sheaf $\mathbb{Q}_X$, earlier results of Hacking [Hac08] show that the cohomology of the link at the vertex of the fan, which is equal to the cohomology of $\mathcal{F}^0$, is related to the top associated graded groups of the weight filtration on the cohomology of $\mathbf{X}$. The latter groups are obtained in terms of a compactification $\overline{\mathbf{X}}$ of the variety $\mathbf{X}$ by a simple normal crossing divisor.

A *tropical compactification*, as introduced by Tevelev [Tev07], is a construction of a compactification of a toric variety with desirable properties. Given any complex subvariety $\mathbf{X}$ of an algebraic torus $\mathbf{T}$ with lattice N, any fan Σ in N gives rise to a toric variety $\mathbb{CP}_\Sigma$, which contains the torus $\mathbf{T}$ and the variety $\mathbf{X}$. This allows one to take the closure $\overline{\mathbf{X}}$ of $\mathbf{X}$ in $\mathbb{CP}_\Sigma$. Moreover, each cone σ of the fan Σ gives rise to a torus orbit $\mathbf{T}^\sigma$, which one may intersect with $\overline{\mathbf{X}}$ to obtain components $\mathbf{X}^\sigma \coloneqq \mathbf{T}^\sigma$, along with their closures $\overline{\mathbf{X}}^\sigma$. If the fan Σ is supported on the tropicalization X, then a result of Tevelev shows that the closure $\overline{\mathbf{X}}$ is compact [Tev07, Proposition 2.3]. Similarly, the tropical variety X admits a closure $\overline{X}$ inside the tropical toric variety $\mathbb{TP}_\Sigma$. There is a similar construction of tropical toric varieties $\mathbb{TP}_\Sigma$, giving a closure $\overline{X}$ of the tropicalization $X \coloneqq \mathrm{trop}(\mathbf{X})$, as well as components X^σ for each cone σ, along with their closures $\overline{X}^\sigma$, see for instance [Kaj08] and [MS15, Chapter 6.2]. Moreover, Tevelev studied for which varieties $\mathbf{X}$ a tropical compactification can be given by a simple normal crossing divisor, calling these *schön* varieties. Earlier, schön hypersurfaces were known as *non-degenerate with respect to their Newton polygon* in the works of Varchenko [Var76a; Var76b] and Kouchnirenko [Kou76].

Thanks to a result of Brion, one may compute the Chow ring $A^\bullet(\mathbb{CP}_\Sigma)$ of a smooth toric variety $\mathbb{CP}_\Sigma$, where we use the notation $R^\bullet$ for a graded ring $\oplus_k R^k$. Taking the cycle class map $\mathrm{cyc}\colon A^\bullet(\mathbb{CP}_\Sigma) \to H^{2\bullet}(\mathbb{CP}_\Sigma)$ from the Chow ring to the cohomology of the variety, and the inclusion $i\colon \overline{\mathbf{X}} \hookrightarrow \mathbb{CP}_\Sigma$ induces the further map $i^*\colon H^{2\bullet}(\mathbb{CP}_\Sigma) \to H^{2\bullet}(\overline{\mathbf{X}})$. Moreover, on the tropical side, there is a similar cycle class map $\mathrm{cyc}\colon A^\bullet(\mathbb{CP}_\Sigma) \to H^{2\bullet}(\mathbb{TP}_\Sigma)$ for the tropical cohomology, which can in turn be composed with the map $H^{2\bullet}(\mathbb{TP}_\Sigma) \to H^{2\bullet}(\overline{X})$. It is a result of Amini and Piquerez that the tropical cycle class map induces an isomorphism $\oplus_k A^k(\mathbb{CP}_\Sigma) \xrightarrow{\sim} \oplus_k H^{k,k}(\overline{X})$ to the central graded part of the tropical cohomology [AP21, Theorem 7.1].

Equipped with the above maps, we proceed to define a partial inverse to the tropical cycle class map by using the isomorphism and mapping all other classes to zero. This leads to a map $\tau^*\colon H^\bullet(\overline{X}) \to H^\bullet(\overline{\mathbf{X}})$ relating tropical cohomology and singular cohomology, and we give the following definition.

Definition 1.3.1 (Definition II.1.1)**.** Let $\mathbf{X} \subseteq \mathbf{T}$ be a subvariety, Σ a unimodular fan with support $X = \mathrm{trop}(\mathbf{X})$, and $\overline{\mathbf{X}}$ and $\overline{X}$ the corresponding compactifications. We say that $\mathbf{X}$ is *cohomologically tropical with respect to* Σ if the induced maps $\tau^*\colon H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ are isomorphisms for all $\sigma \in \Sigma$.

There are two main reasons for demanding τ^* to be an isomorphism for all the strata. First and foremost, it permits a detailed study of which varieties are cohomologically tropical through an inductive argument which we will return to. Secondly, it allows us to extend the arguments from [IKMZ19] by replacing terms in the Steenbrink mixed Hodge structure spectral sequence.

We show that, for schön varieties, the property of being cohomologically tropical does not depend on the choice of underlying unimodular fan for the compactification, provided it is supported on the tropicalization, see Theorem II.4.4. We call a schön subvariety $\mathbf{X} \subseteq \mathbf{T}$ *cohomologically tropical* if it is cohomologically tropical with respect to any choice of unimodular fan supported on the tropicalization.

Seeking a description of which schön varieties are cohomologically tropical, we define a class of varieties which admit strong restrictions on their mixed Hodge structures.

Definition 1.3.2 (Definition II.1.2)**.** A non-singular subvariety $\mathbf{X} \subseteq \mathbf{T}$ of the torus is called *wunderschön with respect to a unimodular fan* Σ with support $\mathrm{trop}(\mathbf{X})$ if all the open strata $\mathbf{X}^\sigma$ of the corresponding compactification $\overline{\mathbf{X}}$ are non-singular and connected, and the mixed Hodge structure on $H^k(\mathbf{X}^\sigma)$ is pure of weight $2k$ for each k.

In particular, wunderschön varieties are schön by applying [Hac08, Lemma 2.7]. As an example, we note that a linear subspace of $\mathbb{CP}^n$, intersected with the central torus $(\mathbb{C}^*)^n$, is wunderschön, see Section II.8.2. An argument using the weight spectral sequence for the mixed Hodge structure shows that the one-dimensional wunderschön varieties are necessarily rational curves punctured in specific points Example II.2.5. We study the properties of wunderschön varieties, showing first that the wunderschön property does not depend on the chosen unimodular fan supported on the tropicalization in Theorem II.4.5. Next, we show that the cohomology of a compactification of a wunderschön variety is divisorial, in the sense that it is generated as a ring by the boundary divisors.

Our main interest in wunderschön varieties stems from their role in the classification of cohomologically tropical varieties. The main theorem of the paper is the following.

Theorem 1.3.3 (Theorem II.6.1)**.** *Let* $\mathbf{X} \subseteq \mathbf{T}$ *be a schön subvariety with tropicalization* $X = \operatorname{trop}(\mathbf{X})$*. Then the following statements are equivalent.*

(1) $\mathbf{X}$ *is wunderschön and* X *is a tropical homology manifold,*

(2) $\mathbf{X}$ *is cohomologically tropical.*

Moreover, if any of these statements holds, then X *is Kähler.*

In Section II.8.2, the above statements are shown to hold for complements of complex hyperplane arrangements considered together with the Bergman fans of their matroids. Showing that these complements are wunderschön is in part done by appealing to work of Shapiro [Sha93] addressing the question of purity of mixed Hodge structures of hyperplane arrangement complements.

This theorem forms an explicit link to the first paper of the thesis, as described in Section 1.2. Indeed, the second condition of (1), i.e. that X is a tropical homology manifold, is one of the main properties investigated in the first paper. The conditions described therein for a fan to be a tropical homology manifold, together with an argument about wunderschön varieties, are used in the third paper of this thesis to show certain varieties are cohomologically tropical.

Equipped with this theorem, we prove the following generalization of the Theorem 1.1.3 of Itenberg, Katzarkov, Mikhalkin and Zharkov.

Theorem 1.3.4 (Theorem II.7.1)**.** *Let* $\pi\colon \mathfrak{X} \to D^*$ *be an algebraic family of subvarieties in* $\mathbb{CP}^n$ *parameterized by the punctured disk and let* $\pi\colon \overline{\mathfrak{X}} \to D$ *be a semistable extension. If the tropicalization* $Z \subseteq \mathbb{TP}^n$ *is a tropical homology manifold and all the open strata in* $\mathfrak{X}_0$ *are wunderschön, then* $H^{p,q}(Z)$ *is isomorphic to the associated graded piece* W_{2p}/W_{2p-1} *of the weight filtration in the limiting mixed Hodge structure* $H^{p+q}_{\lim}$*. The odd weight graded pieces in* $H^{p+q}_{\lim}$ *are all vanishing.*

Moreover, for $t \in D^*$*, we have* $\dim H^{p,q}(\mathfrak{X}_t) = \dim H^{p,q}(Z)$*, for all non-negative integers* p *and* q*.*

Essentially, the main property of tropical manifolds used in the proof is the proof of Theorem 1.1.3 is the local isomorphism between tropical cohomology of the Bergman fan and singular cohomology for a complement of a hyperplane arrangement, which we replace by considering varieties which are merely cohomologically tropical, then use the equivalent description obtained though Theorem II.6.1.

With the above theorems in mind, one may seek to understand which varieties are wunderschön. In dimension one, these are necessarily punctured rational curves, see Example II.2.5, with one puncture for each divisor. In dimension two, the compactification $\overline{\mathbf{X}}$ of a wunderschön variety $\mathbf{X}$ must necessarily be a smooth compact complex surface of irregularity $q = 0$ and geometric genus $p_g = 0$, as studied in detail for instance in [Dol10]. Beyond these simple observations, the classification of wunderschön varieties is open, and progress may yield new examples of cohomologically tropical varieties. Refining the understanding of schön hypersurfaces, i.e. those non-degenerate with respect to their Newton polygon, to an understanding of wunderschön hypersurfaces might be a good first step. We also note that there are some parallels between cohomologically tropical varieties and the *quasilinear tropical compactifications* defined by [Sch23].

Another interesting question is to find direct applications of the more general global Theorem II.7.1, in particular examples where tropical homology manifolds and corresponding wunderschön strata occur naturally.

1.4 Paper three: Axiomatizing curve arrangements

The aim of the third paper of this thesis, Paper III, is to give a concrete family of examples where the properties studied in the two first papers of the thesis are satisfied. We seek to find fans which are tropical homology manifolds, and for which a subclass are wunderschön and consequently also cohomologically tropical.

In the same manner that matroids may be viewed as abstract axiomatizations of hyperplane arrangements, we define *arroids*, which give a possible abstract axiomatization for the incidence geometry of arrangements of curves in the plane. To any arrangement of curves, one may associate an arroid. An arroid $\mathbf{A}$ consists of an underlying set $\mathcal{A}$ where each element i is equipped with a degree d_i, along with a multiset $\mathcal{P}$ of subsets of $\mathcal{A}$. Each set $\mathbf{p} \in \mathcal{P}$ is equipped with a multiplicity function $m_{\mathbf{p}} \colon \mathbf{p}^2 \to \mathbb{Z}$, and the multiset $\mathcal{P}$ must satisfy a Bézout condition in terms of the multiplicity functions. Moreover, when the multiplicity functions are constant taking value one, the arroid is said to be *transversal*. We construct a fan associated to each transversal arroid, and prove the following theorem.

Theorem 1.4.1 (Theorem III.4.6)**.** *For each transversal arroid* $\mathbf{A}$*, there is a fan* $\Sigma_{\mathbf{A}}$*, called the* fan of $\mathbf{A}$*, which is a balanced tropical variety.*

Using transversal arroids, we proceed to study the tropicalization of the complements of very affine transverse arrangements of curves, i.e. containing at least three lines intersecting generically, such that all curves of the arrangement intersect pairwise transversely. In Section III.4.3, we show that for such arrangements, the tropicalization of the complement is computed by the arroid fan.

Theorem 1.4.2 (Theorem III.4.10)**.** *Let* $\mathcal{B}$ *be a transverse very affine arrangement of curves in the plane* $\mathbb{P}^2_K$*. Then the tropicalization* $\operatorname{trop}(X_{\mathcal{B}})$ *of the complement is supported on the fan of the associated transversal arroid* $\mathbf{A}_{\mathcal{B}}$*.*

This recovers that the tropicalization of the complement of a line arrangement is computed using its rank three matroid (see e.g. [MS15, Theorem 4.1.11]), in terms of the Ardila–Klivans fan structure [AK06]. The difficulty in generalizing beyond the transverse case lies primarily in understanding the resolution of singularities that arise when higher order intersections are allowed in the arrangement, as was pointed out in [Cue12, p. 20].

Next, in the spirit of the second paper of this thesis, Paper II, we relate the cohomology of the complement of a curve arrangement with the tropical cohomology of the fan of its arroid.

In Proposition III.6.1, we show that the complement of a *simple* arrangement, i.e. transverse very affine arrangements of lines and conics in $\mathbb{P}^2_{\mathbb{C}}$ with no two intersection points containing exactly the same curves, is wunderschön. This implies that the complements of simple arrangements are cohomologically tropical if and only if the corresponding arroid is a tropical homology manifold by Theorem II.6.1. Using equivalent conditions for an arroid fan to be a tropical homology manifold given in Theorem III.5.4, this yields the following theorem.

Theorem 1.4.3 (Theorem III.6.2)**.** *Let $X_{\mathcal{B}}$ be the complement of a simple arrangement $\mathcal{B}$. Then $X_{\mathcal{B}}$ is cohomologically tropical if and only if the corresponding arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$ is uniquely balanced along each of its rays.*

This characterization relies upon using the unique balancing condition described in Theorem I.4.8. We study which conditions this imposes on curve arrangements in Section III.5.2.

Using Theorem III.6.2, we study the question of *maximality* for a real arrangement and its complexification. Let X be a complex variety defined over $\mathbb{R}$, with $X(\mathbb{R})$ its set of real points and $X(\mathbb{C})$ its set of complex points. The *Smith-Thom inequality* gives bounds for the sum of the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers as follows,

$$b_\bullet(X(\mathbb{R})) \coloneqq \sum_{i\geq 0} b_i(X(\mathbb{R})) \leq \sum_{i\geq 0} b_i(X(\mathbb{C})) =: b_\bullet(X(\mathbb{C})),$$

and the variety is *maximal* if equality is achieved. In [RS23], the authors use a spectral sequence to give bounds on the Betti numbers of real algebraic hypersurfaces arising from Viro's patchworking. By relating tropical homology and Hodge numbers of the complexification using the results of [IKMZ19], Renaudineau and Shaw give bounds for Betti numbers of the real hypersurfaces in terms of the Hodge numbers of their complexifications. This was later generalized to certain higher codimension varieties in [RRS23]. Varieties satisfying these Hodge-number inequalities are called *Hodge expressive* in [BS22], and are used to study moduli spaces of vector bundles on curves. This has led Ambrosi and Manzaroli [AM22] to study the central fiber of a totally real semistable degeneration over a curve, giving conditions on the components of the central fiber for each of the nearby fibers to be Hodge expressive.

Using Theorem III.7.3, as well as the wunderschön and cohomologically tropical properties, we give the following description of certain varieties satisfying the conditions of [AM22].

Theorem 1.4.4 (Theorem III.7.5)**.** *Let $\mathcal{B}$ be a simple arrangement of real curves in $\mathbb{P}^2_{\mathbb{C}}$, with all intersection points being real, and such that the tropicalization*

Trop($X_{\mathcal{B}}$), *which is supported on the arroid fan* $\Sigma_{\mathbf{A}_{\mathcal{B}}}$, *is a tropical homology manifold. Then the following four properties are satisfied:*

(a) $H^i(X_{\mathcal{B}}(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) = 0$ *for* $i \geq 1$,

(b) $X_{\mathcal{B}}$ *is a maximal variety,*

(c) *the mixed Hodge structure on* $H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Q})$ *is pure of type* (i, i) *and* $H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Z})$ *is torsion-free for* $i \geq 1$, *and*

(d) $\dim_{\mathbb{Q}} H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Q}) = \sum_j \dim_{\mathbb{Q}} H^{i,j}(\Sigma_{\mathbf{A}_{\mathcal{B}}})$ *for each* $i \geq 0$.

We construct an infinite family of maximal surfaces in Example III.7.4 satisfying the properties of Theorem III.7.5, which gives examples of the types of variety required in [AM22], using conditions (a), (b) and (c). Moreover, in [RS23], Renaudineau and Shaw study real algebraic hypersurfaces near the tropical limit, giving bounds for Betti numbers in terms of tropical homology, which may be compared to condition (d) above.

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Papers

Paper I

Tropical Poincaré duality spaces

Edvard Aksnes

I

Abstract

The tropical fundamental class of a rational balanced polyhedral fan induces cap products between tropical cohomology and tropical Borel–Moore homology. When all these cap products are isomorphisms, the fan is said to be a *tropical Poincaré duality space*. If all the stars of faces also are such spaces, such as for fans of matroids, the fan is called a *local tropical Poincaré duality space*.

In this article, we first give some necessary conditions for fans to be tropical Poincaré duality spaces and a classification in dimension one. Next, we prove that tropical Poincaré duality for the stars of all faces of dimension greater than zero and a vanishing condition implies tropical Poincaré duality of the fan. This leads to necessary and sufficient conditions for a fan to be a local tropical Poincaré duality space. Finally, we use such fans to show that certain abstract balanced polyhedral spaces satisfy tropical Poincaré duality.

Contents

I.1 Introduction

For an integer $p \geq 0$, a rational polyhedral fan Σ (Definition I.2.2) and a commutative ring R, [IKMZ19] introduced the *tropical homology* $H_\bullet(\Sigma, \mathcal{F}_p^R)$ and *tropical Borel–Moore homology* $H_\bullet^{BM}(\Sigma, \mathcal{F}_p^R)$, along with dual constructions of *tropical cohomology* $H^\bullet(\Sigma, \mathcal{F}_R^p)$ and *tropical cohomology with compact support*

$H_c^\bullet(\Sigma, \mathcal{F}_R^p)$, see Definition I.3.6. These can be computed in many different ways, see e.g. [MZ14; IKMZ19; JSS19; GS23].

The *balancing condition* of tropical geometry (see [BIMS15, Definition 5.8]), can be formulated homologically as the existence of a particular *fundamental class* $[\Sigma, w] \in H_d^{BM}(\Sigma, \mathcal{F}_d^R)$ in tropical Borel–Moore homology ([MZ14, Proposition 4.3], [JRS18, Remark 4.9] and Definition I.3.12), depending on assigning R-valued weights w to maximal faces. One can use the fundamental class to define a *cap product*

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}_R^p) \to H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$$

for all $p, q \in \{0, \dots, d\}$, see [JRS18, Definition 4.11] and Definition I.3.19. If these maps are isomorphisms for all $p, q \in \{0, \dots, d\}$, one says that the fan satisfies *tropical Poincaré duality over R* or is a *tropical Poincaré duality space over R*, see Definition I.4.1. We use the abbreviation TPD for tropical Poincaré duality.

This paper, which generalizes and deepens the results from the author's master's thesis [Aks19], studies two questions related to tropical Poincaré duality over a given commutative ring R.

Question I.1.1. *Which fans satisfy TPD over R?*

The fan of a matroid is a TPD space over $\mathbb{R}$ and $\mathbb{Z}$ by [JSS19, Proposition 4.27] and [JRS18]. Moreover, motivating the question, there are fans satisfying TPD which are not fans of matroids, see Example I.4.4.

A useful property of the cap product is that, for any commutative ring R, when it is non-zero, it is injective (Proposition I.3.23). Using this in the case where R is a field, we can work with Euler characteristics and dimensions of homology groups to give a criterion for a fan to have TPD, under some vanishing assumptions (Proposition I.4.6). Furthermore, we completely classify one-dimensional TPD spaces over an arbitrary commutative ring R.

Theorem I.4.8. *Let R be a commutative ring, and (Σ, w) an R-balanced fan of dimension one. Then (Σ, w) satisfies tropical Poincaré duality over R if and only if it is uniquely R-balanced and all the weights are units in R.*

In Proposition I.4.11, we show that fan tropical hypersurfaces in $\mathbb{R}^n$ must have simplexes as Newton polytopes.

Question I.1.2. *Which fans satisfy TPD over R at each of its faces?*

By this, we mean that for each face $\gamma \in \Sigma$, the *star fan* $\gamma_{\succeq}$ (Definition I.2.6) should be a TPD space over R. We will call this type of fans *local tropical Poincaré duality spaces over R* (Definition I.5.9), which is equivalent to the notion of tropical smoothness defined by Amini and Piquerez [AP21] for $R = \mathbb{Z}$. Fans of matroids can be shown to be local TPD spaces.

Straddling the space between I.1.1 and I.1.2, we prove the following theorem, which shows that when the stars of the faces of a fan are TPD spaces, so is the whole fan, under some vanishing conditions on Borel–Moore homology.

Theorem I.5.4. *Let R be a principal ideal domain, and (Σ, w) be an R-balanced fan of dimension $d \geq 2$, with $H_q^{BM}(\Sigma, \mathcal{F}_p^R) = 0$ for $q \neq d$, for all p. If $(\gamma_{\succeq}, w)$ satisfies TPD over R, for each $\gamma \in \Sigma$ with $\gamma_{\succeq} \neq \Sigma$, then (Σ, w) satisfies TPD over R.*

Noticing the similarity of this result to the conditions for being a local TPD space, we are led to the following characterization of local TPD spaces.

Theorem I.5.10. *Let R be a principal ideal domain, and (Σ, w) a d-dimensional R-balanced fan. Then Σ is a local TPD space over R if and only if $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for all $\gamma \in \Sigma$ and $q \neq d$, and for all faces β of codimension 1, the star fans $\beta_{\succeq}$ are TPD spaces over R.*

In the two-dimensional case, we use Theorem I.5.4 to show that, assuming the vanishing of parts of Borel–Moore homology, a fan is a TPD space if and only if it is a local TPD space, see Proposition I.5.7. This motivates two new questions.

I.5.13 (Geometry of BM homology vanishing)**.** *Let (Σ, w) be an R-balanced d-dimensional fan. Can the fans with $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for each face $\gamma \in \Sigma$, $q \neq d$ and all p be geometrically characterized?*

I.5.14 (Global versus Local TPD)**.** *Let (Σ, w) be an R-balanced fan which satisfies TPD over R. Does $\gamma_{\succeq}$ also satisfy TPD over R for each $\gamma \in \Sigma$?*

In the final part of this paper, we turn to generalizations for *rational polyhedral spaces*, see [JRS18; JSS19], and *abstract tropical R-cycle* (see Definition I.6.1). These can be equipped with tropical homology and cohomology groups, and a balancing condition for abstract tropical R-cycles leads to cap products. *Tropical manifolds* are spaces equipped with charts to Bergman fans of matroids. These are studied in [JRS18; JSS19; GS23], and are shown to satisfy TPD over $\mathbb{R}$ and $\mathbb{Z}$. Thanks to [Yam21], for tropical Calabi-Yau complete intersections, there is a contraction map to an integral affine manifold with singularities (IAMS), relating tropical cohomology and affine homology. For IAMS, there is a cap product map which can be shown to be an isomorphism in certain cases by recent work of [Rud21].

The Mayer–Vietoris arguments used in [JRS18] to show TPD on tropical manifolds can be applied more broadly. We say that an abstract tropical cycle is a *local TPD space over R* if it is built from fans which are local TPD spaces over R. These are the building blocks of the smooth tropical cycles as defined in [AP21]. We then prove the following theorem.

Theorem I.6.5. *Let X be a local tropical Poincaré duality space over R. Then X satisfies tropical Poincaré duality over R.*

Recently, [AP20] establishes a full "Kähler package" for smooth projective tropical cycles, working with rational coefficients. They relate TPD of the canonical compactifications of Bergman fans of matroids to the Poincaré duality of the *Chow ring* of a matroid established in [AHK18], which was used in proving the Heron–Rota–Welsh conjecture. It is suggested in [Huh18] that such "Chow rings" satisfying three properties, collectively dubbed the "Hodge package", should be responsible for the log-concavity of many sequences which arise in mathematics.

In forthcoming work [AAPS23], the authors show that the Tropical Poincaré duality property is a critical ingredient in relating the topology of a variety to the tropical cohomology of its tropicalization.

Organization

In Section I.2, we set conventions for fans, stars and integer weights. Then we define cellular (co)sheaves and cellular (co)sheaf (co)homology.

In Section I.3, we define the tropical multi-tangent cosheaves and sheaves, which we use to define tropical (co)homology. This is used to describe a generalized version of the balancing condition in tropical geometry, to generalize beyond integer weights.

In I.4, we define the TPD over a ring R, and give some necessary conditions. Moreover, we give a complete classification in dimension one, and some criteria in codimension one of $\mathbb{R}^n$ for TPD to hold, which forms a first step towards answering I.1.1.

In Section I.5, we turn to I.1.2. We first relate TPD at the stars of faces to TPD of the whole fan, which is then used to characterize local TPD spaces. We then use our dimension one result to give a more geometric description of the characterization.

Finally, in Section I.6, we use local TPD spaces to construct abstract tropical cycles satisfying tropical Poincaré duality.

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I.2 Preliminaries

In this section, we define and give references to the main objects and concepts used in the remainder of the article. In Section I.2.1, we introduce some conventions for weighted fans and the balancing condition, and for cellular sheaves and cosheaves in Section I.2.2. Finally, we introduce notions of homology and cohomology of cosheaves and sheaves in Section I.2.3.

I.2.1 Cones, fans and stars

Let $N \cong \mathbb{Z}^n$ be a lattice, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ be the associated real vector space.

Definition I.2.1. A *rational polyhedral cone* σ in a lattice N is a set of the form

$$\sigma = \left\{ \sum_{i=1}^{m} a_i v_i \mid a_i \in \mathbb{Z}_{\geq 0} \right\} \subset N$$

for vectors $v_i \in N$, such that $\sigma_{\mathbb{R}} = \sigma \otimes_{\mathbb{Z}} \mathbb{R} \subset N_{\mathbb{R}}$ is closed and strictly convex, hence has a vertex at the origin.

The *lattice* $L_{\mathbb{Z}}(\sigma)$ is the saturated sublattice of N generated by σ, and the *dimension* of a cone is the rank of $L_{\mathbb{Z}}(\sigma)$.

Another cone τ is said to be a *face* of σ if there is some element $m \in \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with $m(x) \geq 0$ for all $x \in \sigma$, i.e. a positive functional, such that $\tau = \{x \in \sigma \mid m(x) = 0\}$. Any face can also be exhibited by setting particular coefficients a_i to 0.

For τ a face of σ, the set $L_{\mathbb{Z}}(\tau) \subset L_{\mathbb{Z}}(\sigma)$ is a sublattice. For $\dim \tau = \dim \sigma - 1$, we may select a *primitive integer vector* $v_{\sigma/\tau} \in N$ such that $L_{\mathbb{Z}}(\sigma) = L_{\mathbb{Z}}(\tau) + \mathbb{Z} v_{\sigma/\tau}$.

Definition I.2.2. A *rational polyhedral fan* Σ is a finite collection of rational polyhedral cones in N such that:

- For any cone $\sigma \in \Sigma$, if τ is a face of σ, then $\tau \in \Sigma$,
- For $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

The cones in Σ are also called *faces*, and the collection of faces of dimension i is denoted by Σ^i. The *dimension of* Σ is the supremum of the dimensions of cones of Σ. We write $\tau \preceq \sigma$ if τ is a face of σ and $\tau \prec \sigma$ if τ is a proper face, which gives a partial ordering on Σ. We say that a face $\sigma \in \Sigma$ is *maximal* if it is maximal with respect to the ordering $\preceq$. We will require that all fans are *pure dimensional* in the sense that all maximal by inclusion faces are of equal dimension.

Abusing notation, we also write Σ for the category associated to the partial ordering $\preceq$, whose objects are the cones $\sigma \in \Sigma$, with a morphism $\tau \to \sigma \in \mathrm{Hom}_{\Sigma}(\tau, \sigma)$ if and only if $\tau \preceq \sigma$.

Note that all cones intersect in a common minimal cell, and since we required each cone to have a vertex, this is the unique vertex v in Σ. Moreover, a rational polyhedral fan corresponds to a *cell complex* in the sense of [She85; Cur14], when considering the fan as glued abstractly from the interiors of the cones.

Example I.2.3. Consider the fan Σ displayed in Figure I.1, which consists of the rays $\tau_1, \tau_2, \tau_3, \tau_4$ and the vertex v. The fan is pure dimensional, its maximal faces are the τ_i and it has dimension 1. It consists of the union of the line $x = 0$ and $y = 0$ when considered in $N_{\mathbb{R}}$. We have $v \prec \tau_i$ for each i.

Example I.2.4. Another example of a fan is shown in Figure I.2. This fan has one vertex v, three one-dimensional cones τ_i, and three two-dimensional cones σ_i. For instance, the faces of σ_1 are the cones τ_1 and τ_2 as well as the vertex v.

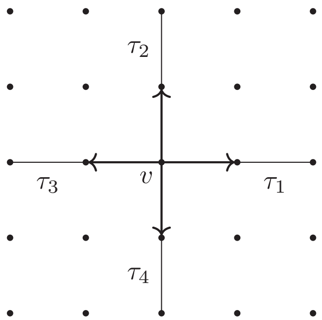

Figure I.1: The cross

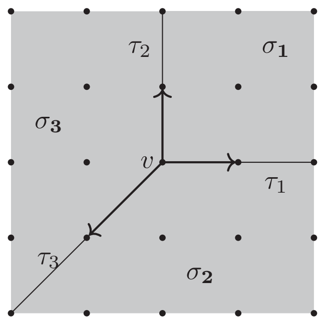

Figure I.2: The complete fan

Fans of particular interest in tropical geometry are the *Bergman fans* of matroids (see [AK06; Zha13] for definitions). These serve as the local models of abstract tropical manifolds (see [MZ14, Section 1.6]).

Example I.2.5. Let M be a matroid on $E = \{0, \dots, n\}$ with lattice of flats $\mathscr{L}$, and let $\mathbb{Z}\{e_0, \dots, e_n\}$ be the lattice of rank $n+1$ generated by elements $e_0, \dots, e_n$. Let N be the quotient defined by

$$0 \longrightarrow \mathbb{Z}\{e_0 + \cdots + e_n\} \longrightarrow \mathbb{Z}\{e_0, \dots, e_n\} \xrightarrow{\pi} N \longrightarrow 0.$$

For any subset $S \subseteq E$, let $p_S = \sum_{i \in S} \pi(e_i)$ in N, so that in particular $p_E = 0$. For any chain $F_\bullet$ of flats of the matroid M,

$$F_\bullet = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\} \subseteq \mathscr{L}.$$

the cone associated to $F_\bullet$ is the non-negative span

$$\sigma(F_\bullet) = \left\{ \sum_{i=1}^{k} a_i p_{F_i} \mid a_i \geq 0,\ i = 1, \dots, k \right\}.$$

The Bergman fan of M is the simplicial fan $\Sigma(M)$ consisting of cones $\sigma(F_\bullet)$ for all flags of flats $F_\bullet$.

The $U_{3,4}$ matroid on the set $E = \{0, \dots, 3\}$ given by the rank function $r \colon 2^E \to \mathbb{Z}_{\geq 0}$ taking values $r(S) = \min(|S|, 3)$ has the lattice of flats given by Figure I.3. The Bergman fan of this matroid is shown in Figure I.4.

Definition I.2.6. The *star* $\gamma_{\succeq}$ at a cone $\gamma \in \Sigma$ is the rational polyhedral fan with underlying set $\cup_{\gamma \preceq \kappa} \widetilde{\kappa} \subset N$, where $\widetilde{\kappa} = \{t(x - y) \mid t \in \mathbb{Z}_{\geq 0}, x \in \kappa, y \in \gamma\} \subseteq N$, subdivided into rational polyhedral cones with a shared vertex.

The *cone* at a face $\gamma \in \Sigma$ is the fan $\gamma_{\preceq}$ consisting of the faces $\kappa \in \Sigma$ for each $\kappa \preceq \gamma$. In particular the vertex v of Σ is the minimal cell in each cone.

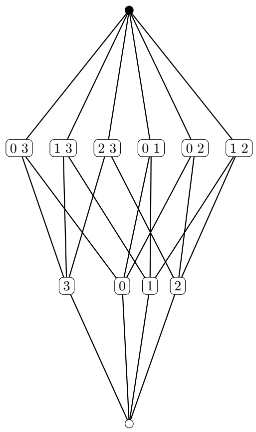

Figure I.3: Lattice of flats of the $U_{3,4}$ matroid.

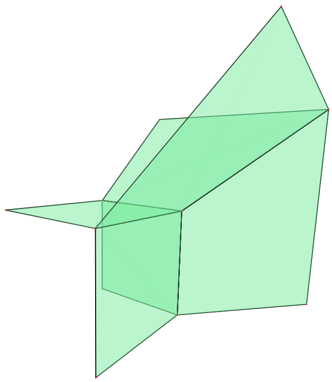

Figure I.4: Bergman fan of the $U_{3,4}$ matroid, visualization by [`polymake`].

Example I.2.7. We give two examples of stars:

I.2.7.1. In the fan from Example I.2.4, the cone τ_1 is contained in the cones σ_1 and σ_2. These give rise to the sets $\widetilde{\sigma_1} = \{(a,b) \in N \mid a \geq 0\}$ and $\widetilde{\sigma_2} = \{(a,b) \in N \mid a \leq 0\}$, so that the star $\tau_{1\succeq}$ has underlying set equal to the whole of N.

I.2.7.2. In the Bergman fan of the $U_{3,4}$ matroid, the star at any of the one-dimensional rays is has underlying set equal to a product of $\mathbb{Z}$ together with the "tropical line", i.e. the fan with rays $(1,1), (-1,0), (0,-1)$ and a vertex at $(0,0)$.

In both cases, these sets must then be cut up so as to form a rational polyhedral fan.

An *integer weight function* on a rational polyhedral fan Σ of dimension d is a function $w\colon \Sigma^d \to \mathbb{Z}$. We are interested in weighted fans satisfying the usual tropical *balancing condition*. This condition is equivalent to being a *Minkowski weight* in the sense of [FS97]. For more on the balancing condition, see for instance [MS15, Definition 3.3.1] or [BIMS15, Definition 5.7].

Definition I.2.8. Let Σ be a rational polyhedral fan of dimension d with weights $w\colon \Sigma^d \to \mathbb{Z}$. We say that Σ together with w is *balanced* at a face $\beta \in \Sigma^{d-1}$ if

$$\sum_{\beta \prec \alpha} w(\alpha) v_{\alpha/\beta} \in L_{\mathbb{Z}}(\beta),$$

using the notation from Definition I.2.1. We say Σ together with w is *balanced* if it is balanced along each face $\beta \in \Sigma^{d-1}$.

Example I.2.9. Our previous examples of fans have all been balanced:

I.2.9.1. The fan of dimension one discussed in Example I.2.3 and shown in Figure I.1 is balanced, for a given weight function $w\colon \Sigma^1 \to \mathbb{Z}$, if and only if $w(\tau_1) = w(\tau_3)$ and $w(\tau_2) = w(\tau_4)$.

I.2.9.2. The fan of dimension two in Example I.2.4 is also balanced if and only if the weight function $w\colon \Sigma^1 \to \mathbb{Z}$ is such that $w(\sigma_1) = w(\sigma_2) = w(\sigma_3)$.

I.2.9.3. It follows from [AK06, Proposition 2], that the stars $\gamma_{\succeq}$ of faces γ in the Bergman fans of a matroid are themselves Bergman fans of matroids. It is shown in [AHK18, Proposition 5.2] that, for the Bergman fan $\Sigma(M)$ of a matroid M, the only weight functions which satisfy the balancing condition are the constant ones. The uniqueness of such a weight function follows from tropical Poincaré duality in [JRS18, Proposition 5.5] and the earlier [Huh14, Theorem 38]. By our later Definition I.3.12, this will mean that these fans are *uniquely $\mathbb{Z}$-balanced*.

I.2.2 Cellular sheaves and cellular cosheaves

One can define *cellular sheaves* and *cellular cosheaves* of modules on a polyhedral fan:

Definition I.2.10. Let R be a commutative ring, Σ a rational polyhedral fan. Then:

- A *cellular R-sheaf* $\mathcal{G}$ is a functor $\mathcal{G}\colon \Sigma \to \mathbf{Mod}_R$.
- A *cellular R-cosheaf* $\mathcal{F}$ is a functor $\mathcal{F}\colon \Sigma^{\mathrm{op}} \to \mathbf{Mod}_R$.

A morphism of sheaves or cosheaves is simply a natural transformation of functors or contravariant functors, respectively.

Remark I.2.11. The category Σ, when viewed as a set, can be given the *Alexandrov topology*, such that cellular sheaves and cosheaves in fact are sheaves and cosheaves with respect to this topology. For more on cellular sheaves and cosheaves, see [Cur14].

We have considered the fan Σ as a category with morphisms $\tau \to \sigma$ whenever τ is a face of σ, so that a sheaf $\mathcal{G}$ induces a map $\mathcal{G}(\tau) \to \mathcal{G}(\sigma)$, and a cosheaf $\mathcal{F}$ induces a map $\mathcal{F}(\sigma) \to \mathcal{F}(\tau)$. This convention is in agreement with [She85; Cur14], but reversed from [Bri97; How08] in the sense that their *sheaves* are our *cosheaves*, and vice versa.

Example I.2.12. Let Σ be a rational polyhedral fan. For a module M over a ring R, the *constant cosheaf M^Σ with values in M* is the cosheaf defined as a functor $M^\Sigma\colon \Sigma^{\mathrm{op}} \to \mathbf{Mod}_R$ taking all objects to M and all morphisms to id_M.

Similarly, the *constant sheaf M_Σ with values in M* is the sheaf defined as a functor $M_\Sigma\colon \Sigma \to \mathbf{Mod}_R$ taking all objects to M and all morphisms to id_M.

I.2.3 Cellular homology and cohomology

Considering the fan Σ as a subset of $N_{\mathbb{R}}$, we select an orientation for each cone $\sigma \in \Sigma$. For each $\tau \prec \sigma$ such that $\dim(\tau) = \dim(\sigma) - 1$, we keep track of the relative orientations by writing $\mathcal{O}(\tau, \sigma) = 1$ if the restriction of the orientation of σ to τ coincides with the orientation of τ, and $\mathcal{O}(\tau, \sigma) = -1$ if it reverses it. In the two next definitions, we use the orientation $\mathcal{O}(\tau, \sigma) = \pm 1$ to construct certain (co)chain complexes for a given (co)sheaf. These definitions are equal to the ones in [She85; Cur14; KSW17], and reversed from [Bri97; How08], who index by codimension.

Definition I.2.13. Given a cellular sheaf $\mathcal{G}$, the *cellular cochain groups* and *cellular cochain groups with compact support* are defined, respectively, by

$$C^q(\Sigma, \mathcal{G}) \coloneqq \bigoplus_{\substack{\sigma \in \Sigma^q \\ \sigma_{\mathbb{R}} \text{ compact}}} \mathcal{G}(\sigma) \quad \text{and} \quad C_c^q(\Sigma, \mathcal{G}) \coloneqq \bigoplus_{\sigma \in \Sigma^q} \mathcal{G}(\sigma),$$

for $q \geq 0$, where $\sigma_{\mathbb{R}}$ is as in Definition I.2.1. The cellular cochain maps

$$d^q \colon C^q(\Sigma, \mathcal{G}) \to C^{q+1}(\Sigma, \mathcal{G}) \quad \text{and} \quad d^q \colon C_c^q(\Sigma, \mathcal{G}) \to C_c^{q+1}(\Sigma, \mathcal{G})$$

are given component-wise for $\tau \in \Sigma^q$ and $\sigma \in \Sigma^{q+1}$ with $\tau \prec \sigma$ by $d_{\tau\sigma} \colon \mathcal{G}(\tau) \to \mathcal{G}(\sigma)$, where

$$d_{\tau\sigma} \coloneqq \mathcal{O}(\tau, \sigma)\mathcal{G}(\tau \to \sigma).$$

If $\tau \not\prec \sigma$, we let the map $d_{\tau\sigma}$ be 0.

The cohomology groups $H^\bullet(\Sigma, \mathcal{G})$ and $H_c^\bullet(\Sigma, \mathcal{G})$ of these complexes are the *cellular sheaf cohomology* and *cellular sheaf cohomology with compact support* with respect to the sheaf $\mathcal{G}$.

Definition I.2.14. Given a cellular cosheaf $\mathcal{F}$, the *cellular chain group* and *Borel–Moore cellular chain groups* are defined, respectively, by

$$C_q(\Sigma, \mathcal{F}) \coloneqq \bigoplus_{\substack{\sigma \in \Sigma^q \\ \sigma_{\mathbb{R}} \text{ compact}}} \mathcal{F}(\sigma) \quad \text{and} \quad C_q^{BM}(\Sigma, \mathcal{F}) \coloneqq \bigoplus_{\sigma \in \Sigma^q} \mathcal{F}(\sigma),$$

for $q \geq 0$, where $\sigma_{\mathbb{R}}$ is as in Definition I.2.1. The cellular chain maps

$$\partial_q \colon C_q(\Sigma, \mathcal{F}) \to C_{q-1}(\Sigma, \mathcal{F}) \quad \text{and} \quad \partial_q \colon C_q^{BM}(\Sigma, \mathcal{F}) \to C_{q-1}^{BM}(\Sigma, \mathcal{F})$$

are given component-wise for $\sigma \in \Sigma^q$ and $\tau \in \Sigma^{q-1}$ by $\partial_{\sigma\tau} \colon \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$, where

$$\partial_{\sigma\tau} \coloneqq \mathcal{O}(\tau, \sigma)\mathcal{F}(\sigma \to \tau).$$

If $\tau \not\prec \sigma$, we let the map $\partial_{\sigma\tau}$ be 0.

The homology groups $H_\bullet(\Sigma, \mathcal{F})$ and $H_\bullet^{BM}(\Sigma, \mathcal{F})$ of these complexes are the *cellular cosheaf homology* and *cellular Borel–Moore cosheaf homology* with respect to $\mathcal{F}$.

Proofs that the cellular (co)chain groups and maps defined above form (co)chain complexes can be found in [Cur14, Definitions 6.2.6-7] and [She85, Theorem 1.1.3].

Remark I.2.15. The above definitions of cellular cohomology work in the more general setting of polyhedral complexes. Since we are working with pointed polyhedral fans, the unique compact cell is the vertex v. Then, for any sheaf $\mathcal{G}$ on a fan Σ, the cellular cochain groups $C^q(\Sigma, \mathcal{G})$ are trivial for $q > 0$, and therefore:

$$H^q(\Sigma, \mathcal{G}) = \begin{cases} \mathcal{G}(v) & \text{for } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for any cosheaf $\mathcal{F}$, the cellular chain groups $C_q(\Sigma, \mathcal{F})$ are trivial for $q > 0$, thus:

$$H_q(\Sigma, \mathcal{F}) = \begin{cases} \mathcal{F}(v) & \text{for } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example I.2.16. Consider the fan from Example I.2.3, with orientations chosen so that $\mathcal{O}(v, \tau_i) = 1$ for all i. The Borel–Moore homology of the constant cosheaf $\mathbb{Z}^\Sigma$ is the homology of the complex

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0,$$

where the matrix ∂_1 is indexed by the τ_i and given by

$$\partial_1 = (\mathcal{O}(v, \tau_i)\, \mathrm{id}_{\mathbb{Z}})_{\tau_i \in \Sigma^1} = (1\ , 1\ , 1\ , 1).$$

The Borel–Moore homology becomes $H_1^{BM}(\Sigma, \mathbb{Z}^\Sigma) = \mathbb{Z}^3$ and $H_0^{BM}(\Sigma, \mathbb{Z}^\Sigma) = 0$.

I.3 Tropical geometry of fans

In this section, we introduce particular cellular (co)sheaves on fans which are of interest in tropical geometry. After examining some properties of the resulting *tropical (co)homology*, we use this to define the *balancing* condition in tropical geometry. Finally, we define the *tropical cap product* associated to a balancing of the fan. We then introduce particular sheaves of interest in tropical geometry. Next, we generalize the balancing condition on fans to weights in arbitrary rings, which finally leads to a treatment of tropical Poincaré duality over arbitrary commutative rings.

I.3.1 Tropical sheaves and cosheaves

For tropical (co)homology, the following sheaves and cosheaves are of interest.

Definition I.3.1 ([IKMZ19, Definition 13])**.** Let Σ be a fan of dimension d in N. For $\sigma \in \Sigma$, let $L_{\mathbb{Z}}(\sigma)$ be the lattice of integer points parallel to the cone σ. For $p = 1, \ldots, d$, the *p-th multi-tangent cosheaf* $\mathcal{F}_p^{\mathbb{Z}}$ is the cellular $\mathbb{Z}$-cosheaf defined by the data:

- For $\sigma \in \Sigma$, one has $\mathcal{F}_p^{\mathbb{Z}}(\sigma) := \sum_{\sigma \preceq \gamma} \bigwedge^p L_{\mathbb{Z}}(\gamma) \subseteq \bigwedge^p N$.
- For $\tau \preceq \sigma$, the morphism $(\sigma \to \tau) \in \mathrm{Hom}_{\Sigma^{\mathrm{op}}}(\sigma, \tau)$ becomes the map $\iota_{\sigma,\tau} \colon \mathcal{F}_p^{\mathbb{Z}}(\sigma) \to \mathcal{F}_p^{\mathbb{Z}}(\tau)$, which is induced by the natural inclusion.

In the $p = 0$ case, we define $\mathcal{F}_0^{\mathbb{Z}} = \mathbb{Z}^{\Sigma}$, with all maps being the identity.

Furthermore, the cellular cosheaf $\mathcal{F}_p^{\mathbb{Z}}$ also gives rise to a cellular sheaf $\mathcal{F}_{\mathbb{Z}}^p$ which is defined by $\mathcal{F}_{\mathbb{Z}}^p(\sigma) := \mathcal{F}_p^{\mathbb{Z}}(\sigma)^*$, with morphisms $\rho_{\tau,\sigma} \colon \mathcal{F}_{\mathbb{Z}}^p(\tau) \to \mathcal{F}_{\mathbb{Z}}^p(\sigma)$ defined by dualizing $\iota_{\sigma,\tau} \colon \mathcal{F}_p^{\mathbb{Z}}(\sigma) \to \mathcal{F}_p^{\mathbb{Z}}(\tau)$.

Finally, following [JRS18], for any ring R, we define a cosheaf $\mathcal{F}_p^R$ by taking the tensor product $\mathcal{F}_p^R(\sigma) = \mathcal{F}_p^{\mathbb{Z}}(\sigma) \otimes_{\mathbb{Z}} R$, giving an R-module, and tensoring the maps as well. Dualizing yields a sheaf $\mathcal{F}_R^p$.

Example I.3.2. We compute some values of these cosheaves:

I.3.2.1. For Example I.2.3, taking the ray τ_1, we have that $\mathcal{F}_1^{\mathbb{Z}}(\tau) = L_{\mathbb{Z}}(\tau) = \langle (1,0) \rangle_{\mathbb{Z}} \subset \mathbb{Z}^2$. For the central vertex v, we have

$$\begin{aligned} \mathcal{F}_1^{\mathbb{Z}}(v) &= \sum_{i=1}^{4} L_{\mathbb{Z}}(\tau_i) \\ &= \langle (1,0) \rangle_{\mathbb{Z}} + \langle (0,1) \rangle_{\mathbb{Z}} + \langle (-1,0) \rangle_{\mathbb{Z}} + \langle (0,-1) \rangle_{\mathbb{Z}} = \mathbb{Z}^2. \end{aligned}$$

The cosheaf $\mathcal{F}_0^{\mathbb{Z}}$ is merely the constant cosheaf taking value $\mathbb{Z}$, so that $\mathcal{F}_0^{\mathbb{Z}}(\tau_i) = \mathbb{Z}$ and $\mathcal{F}_0^{\mathbb{Z}}(v) = \mathbb{Z}$.

I.3.2.2. For Example I.2.4, we have that $\mathcal{F}_2^{\mathbb{Z}}(\sigma_1) = \bigwedge^2 L_{\mathbb{Z}}(\sigma_2) = \langle (1,0) \wedge (0,1) \rangle_{\mathbb{Z}} \cong \mathbb{Z}$.

Remark I.3.3. For any $\mathbb{Z}$-module M and commutative ring R, the product $M_R := M \otimes_{\mathbb{Z}} R$ is an R-module. Moreover, by [Bou98, Proposition III.7.5.8], we have $\bigwedge^p M_R \cong (\bigwedge^p M) \otimes_{\mathbb{Z}} R$. In particular, for $\sigma \in \Sigma$ a maximal face of a d-dimensional fan, $L_R(\sigma) := L_{\mathbb{Z}}(\sigma) \otimes_{\mathbb{Z}} R$ is a free R-module of dimension d, and $\mathcal{F}_R^p(\sigma) = (\bigwedge^p L_{\mathbb{Z}}(\sigma)) \otimes_{\mathbb{Z}} R \cong \bigwedge^p L_R(\sigma)$.

Remark I.3.4. Let Σ be a fan of dimension d. For any $\alpha \in \Sigma^d$, by Remark I.3.3 we have $\mathcal{F}_d^R(\alpha) = \bigwedge^d L_R(\alpha) \cong R$. Given a choice of orientation for α, we can select the unique generator $\Lambda_\alpha \in \mathcal{F}_d^{\mathbb{Z}}(\alpha) = \bigwedge^p L_{\mathbb{Z}}(\sigma)$ compatible with the chosen orientation, and abusing notation, we let $\Lambda_\alpha \in \mathcal{F}_d^R(\alpha) \cong (\bigwedge^p L_{\mathbb{Z}}(\sigma)) \otimes_{\mathbb{Z}} R$ denote the corresponding element $\Lambda_\alpha \otimes 1_R \in \mathcal{F}_d^R(\alpha)$.

Example I.3.5. In Example I.2.4, suppose we choose orientations such that all the one-dimensional rays point outward, with all the two-dimensional cones being oriented clockwise. Choose the standard basis e_1, e_2 for the ambient lattice N. We then have $\Lambda_{\sigma_1} = e_1 \wedge e_2$, $\Lambda_{\sigma_2} = e_1 \wedge e_2$ and $\Lambda_{\sigma_3} = e_1 \wedge e_2$.

Definition I.3.6. The cochain complex $(C^\bullet(\Sigma, \mathcal{F}_R^p), \delta)$ from Definition I.2.13 has cohomology groups $H^q(\Sigma, \mathcal{F}_R^p)$ which are called the (cellular) *tropical cohomology groups with R-coefficients* of Σ. Moreover, the cohomology groups $H_c^q(\Sigma, \mathcal{F}_R^p)$ of the complex $(C_c^\bullet(\Sigma, \mathcal{F}_R^p), \delta)$ are called the (cellular) *compact support tropical cohomology groups with R-coefficients* of Σ.

Similarly, the chain complex $(C_\bullet(\Sigma, \mathcal{F}_p^R), \partial)$ from Definition I.2.14 has homology groups $H_q(\Sigma, \mathcal{F}_p^R)$ which are called the (cellular) *tropical homology groups with R-coefficients* of Σ. Finally, the (cellular) *tropical Borel–Moore homology groups with R-coefficients* $H_q^{BM}(\Sigma, \mathcal{F}_p^R)$ are the homology groups of the chain complex $(C_\bullet^{BM}(\Sigma, \mathcal{F}_p^R), \partial)$.

Proposition I.3.7. *The tropical cohomology with R-coefficients of any fan Σ is*

$$H^q(\Sigma, \mathcal{F}_R^p) = \begin{cases} \mathcal{F}_R^p(v) & \text{for } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $v \in \Sigma$ is the vertex.

Proof. This follows from Remark I.2.15. ∎

Example I.3.8. Consider again the 1-dimensional fan from Example I.2.3. Since it is of dimension 1, the only $\mathcal{F}_R^p$ sheaves are $\mathcal{F}_R^0 \cong R_\Sigma$ and $\mathcal{F}_R^1$. By Proposition I.3.7, the only non-zero cohomology groups are $H^0(\Sigma, \mathcal{F}_R^0) = R$ and $H^0(\Sigma, \mathcal{F}_R^1) = \mathcal{F}_R^1(v) = R^2$.

Similarly, the only $\mathcal{F}_p^R$ cosheaves are $\mathcal{F}_0^R \cong R^\Sigma$ and $\mathcal{F}_1^R$. The computation of the homology with the constant cosheaf $\mathbb{Z}^\Sigma$ given in Example I.2.16 carries through to R^Σ, giving $H_1^{BM}(\Sigma, \mathcal{F}_0^R) = R^3$ and $H_0^{BM}(\Sigma, \mathcal{F}_0^R) = 0$. Finally, to compute the Borel–Moore homology for $\mathcal{F}_1^R$, we have the chain complex

$$0 \longrightarrow \oplus_{\tau_i \in \Sigma^1} \mathcal{F}_1^R(\tau_i) \xrightarrow{\partial_1} \mathcal{F}_1^R(v) \longrightarrow 0.$$

Selecting the $\mathbb{Z}$-basis $e_1 = (1,0)$, $e_2 = (0,1)$ for N, we can write this complex as

$$0 \longrightarrow \langle (1,0) \rangle \oplus \langle (0,1) \rangle \oplus \langle (-1,0) \rangle \oplus \langle (0,-1) \rangle \xrightarrow{\partial_1} \langle (1,0), (0,1) \rangle \longrightarrow 0,$$

where ∂_1 is now the direct sum of the inclusion maps, and everything is suitably tensored with R. The Borel–Moore homology can then be shown to be given by $H_0^{BM}(\Sigma, \mathcal{F}_1^R) = 0$ and $H_1^{BM}(\Sigma, \mathcal{F}_1^R) = \langle (a,b,a,b) \mid a, b \in R \rangle \cong R^2$.

Example I.3.9. We now show how to perform the above computations using the [KSW17] package for [`polymake`], when working with rational coefficients. A code example is given in Figure I.5. To compute with [`polymake`], one specifies the rays of a fan, as well as which rays form a cone. The fan must be input in projective coordinates, so that there is a distinct projection point $[1,0,0]$, with all rays expressed using an embedding of N into the hyperplane $H = \{(x_0, x_1, x_2) \mid x_0 = 0\}$. Thus the ray τ_1 is $[0,1,0]$. Similarly the cones must all be given as including the projection point $[1, 0, 0]$, so that the one-dimensional ray τ_2 is given as $[0, 2]$.

```
application 'fan';
$fan = new fan::PolyhedralFan(
    INPUT_RAYS=>[
        [1,0,0],[0,1,0],[0,0,1],[0,-1,0],[0,0,-1]
    ],
    INPUT_CONES =>[
        [0,1],[0,2],[0,3],[0,4],
    ]
);
$complex = new fan::PolyhedralComplex($fan);
$dim = $complex -> AMBIENT_DIM;

@cohom_dimensions = ();
@BM_dimensions = ();
for(my $i=0;$i<$dim;$i++){
    my $fi = $complex->fcosheaf($i);
    my $si=$complex->usual_chain_complex($fi);
    my $bmi=$complex->borel_moore_complex($fi);
    push @cohom_dimensions, topaz::betti_numbers($si);
    push @BM_dimensions, topaz::betti_numbers($bmi);
}

print "H^q(Sigma, F^p) dimensions; q=columns, p=rows";
print join("\n", @cohom_dimensions),"\n\n";
print "H_q^{BM}(Sigma, F_p) dimensions; q=columns, p=rows";
print join("\n", @BM_dimensions),"\n\n";
```

Figure I.5: Code in [`polymake`] to compute the tropical cohomology and Borel–Moore homology in Example I.2.3

Note also that one may use the command `$complex -> VISUAL;` to receive a visualisation for two- and three-dimensional fans. The output of the code in Figure I.5 is shown in Figure I.6:

```
H^q(Sigma, F^p) dimensions; q=columns, p=rows
1 0
2 0

H_q^{BM}(Sigma, F_p) dimensions; q=columns, p=rows
0 3
0 2
```

Figure I.6: Output from Figure I.5

Recall from Definition I.2.6 that one must subdivide the stars $\gamma_{\succeq}$ of faces $\gamma \in \Sigma$ to obtain a fan structure. The next proposition shows that the tropical cohomology of $\gamma_{\succeq}$ is determined directly by $\mathcal{F}_p^R(\gamma)$, and that the tropical Borel–Moore homology can be computed using a simpler complex than the one coming from the subdivision.

Proposition I.3.10. *Let Σ be a fan and $\gamma \in \Sigma$ a face of dimension r. Let $\mathcal{F}^p_{R,\Sigma}$ and $\mathcal{F}^p_{R,\gamma}$ denote the p-th multi-tangent sheaves on Σ and $\gamma_{\succeq}$ respectively. Then*

$$H^0(\gamma_{\succeq}, \mathcal{F}^p_{R,\gamma}) \cong \mathcal{F}^p_{R,\Sigma}(\gamma).$$

Similarly, let $\mathcal{F}_p^{R,\Sigma}$ and $\mathcal{F}_p^{R,\gamma}$ denote the p-th multi-tangent cosheaves on Σ and $\gamma_{\succeq}$ respectively. Then the Borel–Moore homology $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^{R,\gamma})$ is isomorphic to the homology of the complex

$$0 \longrightarrow \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\gamma}} \mathcal{F}_p^{R,\Sigma}(\alpha) \overset{\partial_q|_{\alpha\succ\gamma}}{\longrightarrow} \cdots \longrightarrow \bigoplus_{\substack{\kappa\in\Sigma^{r+1} \\ \kappa\succ\gamma}} \mathcal{F}_p^{R,\Sigma}(\kappa) \overset{\partial_q|_{\kappa\succ\gamma}}{\longrightarrow} \mathcal{F}_p^{R,\Sigma}(\gamma) \longrightarrow 0,$$

where we define $\partial_q^\gamma = \oplus \partial_{\sigma\tau}$ with the sum taken over all $\sigma, \tau \succeq \gamma$, $\sigma \in \Sigma^q$ and $\tau \in \Sigma^{q-1}$.

Proof. First, by Definition I.2.6, we must choose a subdivision of the space with support given by the cones $\widetilde{\kappa} = \{t(x-y) \mid t \geq 0, x \in \kappa, y \in \gamma\}$ for each $\kappa \succeq \gamma$, and we will have only one compact cell given by the created vertex $\widetilde{v} \in \gamma_{\succeq}$. By Remark I.2.15, we have $H^0(\gamma_{\succeq}, \mathcal{F}^p_{R,\gamma}) = \mathcal{F}^p_{R,\gamma}(\widetilde{v})$. Then, observe that for each $\kappa \succ \gamma$, the lattice is unchanged in the sense that $L_{\mathbb{Z}}(\widetilde{\kappa}) = L_{\mathbb{Z}}(\kappa)$ as subspaces of N, and each maximal dimensional face α in the subdivision of $\gamma_{\succeq}$ is a subspace of a $\widetilde{\kappa}$. In particular, this implies that $\mathcal{F}^p_{R,\gamma}(\widetilde{v}) \cong \mathcal{F}^p_{R,\Sigma}(\gamma)$.

Next, observe that the Borel–Moore homology is the equal to the regular homology of the fan when seen as a subset of the one-point compactification $N \cup \{\infty\}$ of the ambient lattice N. Then every cone $\sigma \in \gamma_{\succeq}$ becomes a disk σ_∞ in $N \cup \{\infty\}$, and we have a CW complex structure on the underlying set of $\gamma_{\succeq} \cup \{\infty\}$. Then, similarly to [MZ14, Section 2.2] and [JRS18, Remark 2.8], note that:

$$C_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = \oplus_{\sigma\in\gamma^q_{\succeq}} \mathcal{F}_p^R(\sigma) = \oplus_{\sigma\in\gamma^q_{\succeq}} H_q(\sigma_\infty, \partial(\sigma_\infty), \mathcal{F}_p^R(\sigma)),$$

where the right hand side becomes the cellular homology with coefficients in the local system induced by $\mathcal{F}_p^R$ of the CW complex $\gamma_{\succeq} \cup \{\infty\}$. This can be computed with an arbitrary CW structure. Therefore, the given complex, which is the local system homology of the CW structure induced on $|\gamma_{\succeq} \cup \{\infty\}|$ by taking the non-subdivided cell structure of $\gamma_{\succeq}$ will compute the Borel–Moore homology of the $\mathcal{F}_p^R$ cosheaf. ■

The above proposition shows that, when working with stars of faces in a fan, the particular cellular structure does not change the tropical (co)homology. This is not the case for general (co)sheaves, see for instance the *wave tangent sheaves* defined in [MZ14, Section 3].

I.3.2 Balancing in tropical geometry

It was observed in [JRS18, Remark 4.9] and [MZ14, Proposition 4.3] that the balancing condition from Definition I.2.8 can be equivalently formulated as the condition that a particular tropical Borel–Moore chain is closed. In this subsection, we use this observation to form a generalization of the balancing condition to arbitrary commutative rings.

Definition I.3.11. Let R be a ring. An *R-weight function* $w\colon \Sigma^d \to R$ on a d-dimensional fan Σ is a function such that for all $\alpha \in \Sigma^d$, the weight $w(\alpha)$ is not a zero-divisor. A pair (Σ, w) will be called an *R-weighted fan*.

Definition I.3.12. An R-weighted fan (Σ, w) is *R-balanced* if *the fundamental chain* $\mathrm{Ch}(\Sigma, w)$ given by

$$\mathrm{Ch}(\Sigma, w) \coloneqq (w(\alpha)\Lambda_\alpha)_{\alpha\in\Sigma^d} \in C_d^{BM}(\Sigma, \mathcal{F}_d)$$

is a cycle, where the Λ_α are chosen as in Remark I.3.4. In this case, we have $H_d^{BM}(\Sigma, \mathcal{F}_d^R) \neq 0$, together with an induced *fundamental class*

$$[\Sigma, w] = [\mathrm{Ch}(\Sigma, w)] \in H_d^{BM}(\Sigma, \mathcal{F}_d^R).$$

If $H_d^{BM}(\Sigma, \mathcal{F}_d^R) = \langle[\Sigma, w]\rangle \cong R$, we say that Σ is *uniquely R-balanced* by w.

Example I.3.13. We now compute the fundamental chain in the fan Σ of Example I.2.4. Choose orientations such that the elements Λ_{σ_i} are as in Example I.3.5. Moreover, we choose a weight function assigning the value 1 to each of the cones σ_i. Then the fundamental chain is:

$$(\Lambda_{\sigma_i})_{i=1}^3 = (e_1 \wedge e_2, e_1 \wedge e_2, e_1 \wedge e_2) \in C_2^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Z}}).$$

It is then straightforward to check that, under the boundary map ∂_2, taking into account orientations, this chain is mapped to zero. For instance, for the component of $C_1^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Z}})$ corresponding to τ_1, we have

$$-e_1 \wedge e_2 + e_1 \wedge e_2 = 0 \in \mathcal{F}_2^{\mathbb{Z}}(\tau_1),$$

with the first 2-wedge corresponding to σ_1 and the second to σ_2. Thus the fan is $\mathbb{Z}$-balanced, and there is a fundamental class $[\Sigma, w] \in H_2^{BM}(\Sigma, \mathcal{F}_2^R)$. Moreover, it can be checked that this class generated the whole cohomology module, so that this fan is in fact uniquely R-balanced.

The above definition, which is equivalent to the usual balancing condition [BIMS15, Definition 5.8], is similar in flavor to the description given by [BH17, Theorem 2.9]. We illustrate this with the following example.

Example I.3.14. In this example, we explicitly relate the above definition of balancing to the one given in Definition I.2.8. Let (Σ, w) be a $\mathbb{Z}$-balanced fan of dimension d in the sense of Definition I.3.12. Then, for each $\beta \in \Sigma^{d-1}$, we pick a generator $\Lambda_\beta \in L_{\mathbb{Z}}(\beta)$ respecting the orientation. Now for each $\alpha \in \Sigma^d$ with $\beta \prec \alpha$, the vector $v_{\alpha/\beta}$ from Definition I.2.1 is such that $\Lambda_\alpha = \Lambda_\beta \wedge v_{\alpha/\beta}$.

Looking at the β-component of $\partial\colon C_d^{BM}(\Sigma, \mathcal{F}_d^{\mathbb{Z}}) \to C_{d-1}^{BM}(\Sigma, \mathcal{F}_d^{\mathbb{Z}})$, we have

$$\partial((w(\alpha)\Lambda_\alpha))_\beta = \sum_{\beta\prec\alpha} w(\alpha)\Lambda_\alpha = \Lambda_\beta \wedge \sum_{\beta\prec\alpha} w(\alpha)v_{\alpha/\beta}.$$

Therefore, the chain $\mathrm{Ch}(\Sigma, w)$ is closed if and only if each of the faces β are balanced in the sense of Definition I.2.8. Thus the definitions are equivalent.

Proposition I.3.15. *Let (Σ, w) be an R-balanced fan and $\gamma \in \Sigma$ a face. Then $(\gamma_{\succeq}, w)$ is R-balanced, where w is understood to be the weight function induced on $\gamma_{\succeq}$ by w.*

Proof. By Proposition I.3.10, we have that $H_d^{BM}(\gamma_{\succeq}, \mathcal{F}_d^R)$ can be viewed as the kernel of the map

$$\bigoplus_{\substack{\alpha \in \Sigma^d \\ \alpha \succ \gamma}} \mathcal{F}_d^R(\alpha) \xrightarrow{\partial_d^\gamma} \bigoplus_{\substack{\beta \in \Sigma^{d-1} \\ \beta \succ \gamma}} \mathcal{F}_d^R(\beta).$$

Moreover, the class

$$\mathrm{Ch}(\gamma_{\succeq}, w) = (w(\alpha)\Lambda_\alpha)_{\substack{\alpha \in \Sigma^d \\ \alpha \succ \gamma}}$$

is a cycle since $\mathrm{Ch}(\Sigma, w)$ is a cycle in $C_d^{BM}(\Sigma, \mathcal{F}_d^R)$. Thus $(\gamma_{\succeq}, w)$ is R-balanced and we have a fundamental class $[\gamma_{\succeq}, w] \in H_d^{BM}(\gamma_{\succeq}, \mathcal{F}_d^R)$. ∎

I.3.3 Tropical cap product

There is a cap product relating $H^q(\Sigma, \mathcal{F}_R^p)$ to $H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$, which will be at the core of tropical Poincaré duality. We extend the version given in [JRS18, Definition 4.10] for $R = \mathbb{Z}$ to arbitrary commutative rings R, using the contraction map from multilinear algebra for a general ring R, as developed in [Bou98, Section III.11].

Definition I.3.16 ([Bou98, Section III.11.9])**.** Let M_R be a rank d free R-module, M_R^* the dual module, and $0 \leq p_1 \leq p_2 \leq d$. The *interior product* or *contraction* defined by $y = y_1 \wedge \cdots \wedge y_{p_2} \in \bigwedge^{p_2} M_R$ is the map

$$\lrcorner\, y \colon \bigwedge^{p_1} M_R^* \to \bigwedge^{p_2 - p_1} M_R,$$

which is defined on $x = x_1 \wedge \cdots \wedge x_p \in \bigwedge^{p_1} M_R^*$ to be

$$x \lrcorner\, y = (-1)^{p_1(p_1-1)/2} \sum_a \mathrm{sign}(a) \left(\prod_{i=1}^{p_1} x_i(y_{a(i)}) \right) y_{a(p+1)} \wedge \cdots \wedge y_{a(d)},$$

where the sum is taken over all permutations $a \in S_d$ which are increasing on $1, \ldots, p$ and $p+1, \ldots, d$, and is extended linearly.

Remark I.3.17. In [Bou98, Section III.11.10], an explicit formula for this contraction map is given in terms of bases. Letting $e_1, \ldots, e_m$ be a basis of M_R, the elements $e_I \coloneqq e_{i_1} \wedge \cdots \wedge e_{i_{p_2}} \in \bigwedge^{p_2} M_R$, for all $I = \{i_1 < \cdots < i_{p_2}\} \subseteq [m]$ ordered strictly increasing subsets of size p, form a basis of $\bigwedge^{p_2} M_R$. Letting $f_1, \ldots, f_m$ be the dual basis to the e_i for M_R^*, the elements f_J, for all

$J = \{j_1 < \cdots < j_{p_1}\} \subseteq [m]$, form a basis of $\bigwedge^{p_1} M_R^*$. Then the contraction map defined by e_J is given by

$$\begin{cases} f_K \lrcorner e_J = 0 & \text{if } K \not\subseteq J, \\ f_K \lrcorner e_J = (-1)^{v+p_1(p_1-1)/2}\, e_{J\smallsetminus K} & \text{if } K \subseteq J \text{ and } p_1 = |K|, \end{cases}$$

where v is the number of ordered pairs $(\lambda, \mu) \in K \times (J \smallsetminus K)$ such that $\lambda > \mu$.

A proof that these contraction maps are the same as the formulation in terms of compositions given in [Aks19; AP21] follows from the arguments given in [Aks19, Lemma 4.1.4].

Definition I.3.18. For each facet $\alpha \in \Sigma^d$ of a d-dimensional R-balanced fan Σ, we have chosen a generator $\Lambda_\alpha \in \mathcal{F}_d^R(\alpha) = \bigwedge^d L_R(\alpha) \cong R$ by Remark I.3.4, and a weight $w(\alpha) \in R$ which is not a zero-divisor. The *contraction* defined by $w(\alpha)\Lambda_\alpha$ is the map

$$\lrcorner\, w(\alpha)\Lambda_\alpha \colon \bigwedge^p L_R(\alpha)^* \to \bigwedge^{d-p} L_R(\alpha).$$

Since $w(\alpha)$ is not a zero-divisor, Remark I.3.17 shows that this map is injective. It is an isomorphism if and only if $w(\alpha)$ is a unit.

Definition I.3.19. Let the weighted fan (Σ, w) be an R-balanced fan of dimension d. The *cap product* $\frown \mathrm{Ch}(\Sigma, w)$ with the fundamental chain of Σ is the map given by:

$$\begin{aligned} \frown \mathrm{Ch}(\Sigma, w) \colon C^q(\Sigma, \mathcal{F}_R^p) &\to C_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R) \\ (u_\gamma)_{\gamma\in\Sigma^q} &\mapsto \left(\sum_{\substack{\alpha\in\Sigma^d \\ \gamma,\tau\preceq\alpha}} w(\alpha)\iota_{\alpha,\tau}\left(\rho_{\gamma,\alpha}(u_\gamma) \lrcorner \Lambda_\alpha\right) \right)_{\tau\in\Sigma^{d-q}} \end{aligned}$$

where Λ_α is as in Remark I.3.4.

Remark I.3.20. For any chain $\sigma \in C_q^{BM}(\Sigma, \mathcal{F}_p^R)$, a cap product $\frown \sigma$ can be similarly defined. It is noted in [JRS18, p. 13] that the Leibniz formula holds for these cap product, such that $\partial(\alpha \frown \sigma) = (-1)^{q+1}(d(\alpha) \frown \sigma - \alpha \frown \partial(\sigma))$. In the case where $R = \mathbb{R}$, the Leibniz formula also follows from [JSS19, Remark 2.2, Definition 4.11]. Therefore the above defined map descends to tropical (co)homology, and we have the *cap product with the fundamental class* $\frown [\Sigma, w]$

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}_R^p) \to H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R).$$

Example I.3.21. In Example I.3.14, we computed the fundamental class of the fan Σ from Example I.2.4, given the all-one weight function w. We will now compute some examples of the cap product. Let $e_1, e_2 \in N \cong \mathbb{Z}^2$ be the standard basis for the underlying lattice, and e_1^*, e_2^* the dual basis for the dual lattice N^*. Then

$$\begin{aligned} H^0(\Sigma, \mathcal{F}^0_{\mathbb{Z}}) &= \mathcal{F}^0_{\mathbb{Z}}(v) = \mathbb{Z}, \\ H^0(\Sigma, \mathcal{F}^1_{\mathbb{Z}}) &= \mathcal{F}^1_{\mathbb{Z}}(v) = \langle e_1^*, e_2^* \rangle_{\mathbb{Z}}, \\ H^0(\Sigma, \mathcal{F}^2_{\mathbb{Z}}) &= \mathcal{F}^2_{\mathbb{Z}}(v) = \langle e_1^* \wedge e_2^* \rangle_{\mathbb{Z}}. \end{aligned}$$

Moreover, all other cohomology groups are zero, by Proposition I.3.7. Next, the Borel–Moore chain complexes are given by

$$\begin{aligned} &C^{BM}_\bullet(\Sigma, \mathcal{F}_0^{\mathbb{Z}})\colon \quad 0 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0, \\ &C^{BM}_\bullet(\Sigma, \mathcal{F}_1^{\mathbb{Z}})\colon \quad 0 \to \bigoplus_{i=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\sigma_i) \to \bigoplus_{j=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\tau_j) \to \mathcal{F}_1^{\mathbb{Z}}(v) \to 0, \\ &C^{BM}_\bullet(\Sigma, \mathcal{F}_2^{\mathbb{Z}})\colon \quad 0 \to \bigoplus_{i=1}^3 \mathcal{F}_2^{\mathbb{Z}}(\sigma_i) \to \bigoplus_{j=1}^3 \mathcal{F}_2^{\mathbb{Z}}(\tau_j) \to \mathcal{F}_2^{\mathbb{Z}}(v) \to 0. \end{aligned}$$

We now compute $H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$ as an example. We have that $\mathcal{F}_1^{\mathbb{Z}}(\tau_j) = \langle e_1, e_2 \rangle_{\mathbb{Z}}$ for each j, and $\mathcal{F}_1^{\mathbb{Z}}(\sigma_i) = \langle e_1, e_2 \rangle_{\mathbb{Z}}$ for each i. Taking the direct sum of these bases in $\bigoplus_{i=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\sigma_i)$, and respecting the orientations, we may express the differential $\partial_2 : \bigoplus_{i=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\sigma_i) \to \bigoplus_{j=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\tau_j)$ in coordinates as:

$$\begin{aligned} (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3) \mapsto (&-\alpha^1 + \alpha^2, -\beta^1 + \beta^2, \alpha^1 - \alpha^3, \\ &\beta^1 - \beta^3, -\alpha^2 + \alpha^3, -\beta^2 + \beta^3), \end{aligned}$$

with α^i and β^i corresponding respectively to e_1 and e_2 for σ_i. We compute a basis for the kernel of this map, i.e. a basis for $H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$ to be:

$$H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = \langle (1,0,1,0,1,0), (0,1,0,1,0,1) \rangle_{\mathbb{Z}} \subset C_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$$

Similar computations show that the remaining Borel–Moore homology groups are given by:

$$\begin{array}{lll} H_0^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) = 0 & H_1^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) = 0 & H_2^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) \cong \mathbb{Z} \\ H_0^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = 0 & H_1^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = 0 & H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) \cong \mathbb{Z}^2 \\ H_0^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = 0 & H_1^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = 0 & H_2^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Z}}) = \langle [\Sigma, w] \rangle \cong \mathbb{Z}. \end{array}$$

Finally, we now compute an example for the cap product map, in particular $\frown [\Sigma, w]\colon H^0(\Sigma, \mathcal{F}^1_{\mathbb{Z}}) \to H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$. Working from the definition, we have that

$$e_1^* \mapsto (e_1^* \lrcorner \Lambda_{\sigma_1}, e_1^* \lrcorner \Lambda_{\sigma_2}, e_1^* \lrcorner \Lambda_{\sigma_3}) \in \bigoplus_{i=1}^3 \mathcal{F}_1^{\mathbb{Z}}(\sigma_i).$$

Expanding this using the (α^i, β^i) basis from above, these contractions are such that $e_1^* \mapsto (1,0,1,0,1,0)$, and one can similarly check that $e_2^* \mapsto (0,1,0,1,0,1)$. This shows that $\frown [\Sigma, w]$ is in this case an isomorphim.

Proposition I.3.22. *Let (Σ, w) be an R-balanced fan of dimension d. The cap product with the fundamental class $\frown [\Sigma, w]$ in tropical cohomology*

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}_R^p) \to H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$$

is the 0-map for $q \neq 0$.

Proof. By Proposition I.3.7, we have that $H^q(\Sigma, \mathcal{F}_p^R) = 0$ for $q \neq 0$, hence this cap product is only non-trivial when $q \neq 0$, ■

The above proposition shows that in the fan-case, the only interesting cap products are of the form $\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}_R^p) \to H_d^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$, for $p = 0, \dots, d$. Moreover, in Proposition I.3.23 below, we show that these are injective for any commutative ring R. In the case where $R = \mathbb{R}$, this was shown in [Aks19, Theorem 4.3.1], and for $R = \mathbb{Z}$, it is stated in [AP21, Section 3.2.2].

Proposition I.3.23. *For an R-balanced fan (Σ, w) of dimension d, the map*

$$\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}_R^p) \to H_d^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$$

is injective.

Proof. We have that $H_d^{BM}(\Sigma, \mathcal{F}_{d-p}^R) = \ker(\partial_d)$, and $H^0(\Sigma, \mathcal{F}_R^p) = \mathcal{F}_R^p(v)$, so that $\frown [\Sigma, w]$ is exactly

$$\begin{aligned} \frown \mathrm{Ch}(\Sigma, w) \colon \mathcal{F}_R^p(v) &\to \bigoplus_{\alpha \in \Sigma^d} \mathcal{F}_{d-p}^R(\alpha) \\ u &\mapsto (\rho_{v,\alpha}(u) \lrcorner\, w(\alpha)\Lambda_\alpha)_{\alpha \in \Sigma^d} \end{aligned}$$

where the image lies in $H_d^{BM}(\Sigma, \mathcal{F}_{d-p}^R) \subseteq \bigoplus_{\alpha \in \Sigma^d} \mathcal{F}_{d-p}^R(\alpha)$. This is the composition of the map $\oplus_\alpha \rho_{v,\alpha} \colon \mathcal{F}_R^p(v) \to \oplus_{\alpha \in \Sigma^d} \mathcal{F}_R^p(\alpha)$, which is injective, since it is dual to the surjection $\oplus_{\alpha \in \Sigma^d} \mathcal{F}_p^R(\alpha) \to \mathcal{F}_p^R(v)$, and the direct sum of the contractions $\oplus_{\alpha \in \Sigma^d} \lrcorner\, w(\alpha)\Lambda_\alpha$, which are injective (Definition I.3.18). Thus this cap product is the composition of injective maps and is therefore injective. ■

Proposition I.3.24. *Let (Σ, w) be an R-balanced fan and $\gamma \in \Sigma$ a face. Then the cap product map on the star fan*

$$\frown [\gamma_{\succeq}, w] \colon H^0(\gamma_{\succeq}, \mathcal{F}_R^p) \to H_d^{BM}(\gamma_{\succeq}, \mathcal{F}_{d-p}^R)$$

is given by

$$u \mapsto (u \lrcorner\, w(\alpha)\Lambda_\alpha)_{\substack{\alpha \in \Sigma^d, \\ \alpha \succ \gamma}}$$

where we identify $H^0(\gamma_{\succeq}, \mathcal{F}_R^p) \cong \mathcal{F}_p^R(\gamma)$, and $H_d^{BM}(\gamma_{\succeq}, \mathcal{F}_{d-p}^R)$ as the kernel of the first map in the complex from Proposition I.3.10.

Proof. The identifications are justified by Proposition I.3.10, and the existence of this fundamental class by Proposition I.3.15. It remains to show that the stated formula corresponds to the cap product.

Consider a subdivision making $\gamma_{\succeq}$ a pointed fan. Each d-cell $\widetilde{\alpha}$ of the subdivision maps to a d-cell $\alpha \succ \gamma$ of Σ, similarly to the proof of Proposition I.3.10. The formula then follows from the induced map in homology. ■

I.4 Tropical Poincaré duality

In Section I.4.1, we define TPD over a ring R, and give an example of a non-matroidal fan satisfying the duality. In Section I.4.2, we give some necessary conditions for the duality to hold, along with a characterization by an Euler characteristic condition. Finally, in Section I.4.3, we turn to the problem of determining which fans are TPD spaces. We classify all the one-dimensional fans satisfying TPD over a ring R and study tropical fan hypersurfaces in $\mathbb{R}^n$ satisfying TPD. This forms a first step towards answering I.1.1.

I.4.1 Definition and examples

In this subsection, we define what it means for a fan to satisfy TPD over a commutative ring R. When $R = \mathbb{Z}$, this is the definition from [JRS18, Definition 5.2], and when $R = \mathbb{R}$, our definition can be shown to be equivalent to [JSS19, Definition 4.12].

Definition I.4.1. We say that an R-balanced rational polyhedral fan Σ of dimension d with weights w satisfies *tropical Poincaré duality over R* if the cap product with the fundamental class

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}_R^p) \to H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R)$$

is an isomorphism for all $p, q = 0, \ldots, d$.

Example I.4.2. Returning again to Example I.2.4, one can verify that all the possible cap products are isomorphisms, as we did explicitly in Example I.3.21 for the cap product $\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}_{\mathbb{Z}}^1) \to H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$, so that this fan satisfies tropical Poincaré duality over $\mathbb{Z}$.

Example I.4.3. Similarly, explicit computations can be carried out for Example I.2.3. Comparing back to Example I.3.8, we have that $\dim_{\mathbb{Z}} H_1^{BM}(\Sigma, \mathcal{F}_1^R) = 2$ and $\dim_{\mathbb{Z}} H_1^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) = 3$, while $\dim_{\mathbb{Z}} H^0(\Sigma, \mathcal{F}_1^R) = 2$ and $\dim_{\mathbb{Z}} H^0(\Sigma, \mathcal{F}_0^R) = 1$.

Thus the cap product maps

$$\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}_{\mathbb{Z}}^0) \to H_1^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}), \quad \text{and}$$
$$\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}_{\mathbb{Z}}^1) \to H_1^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}})$$

are not isomorphisms, and the fan does not satisfy tropical Poincaré duality over $\mathbb{Z}$.

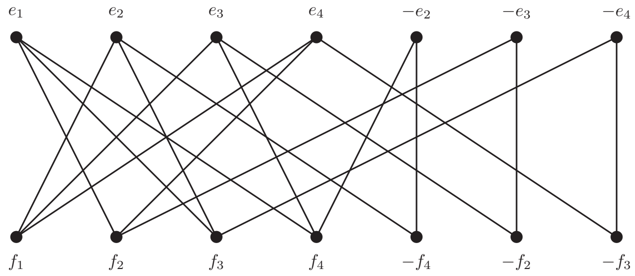

Figure I.7: The graph of cones for Example I.4.4.

As mentioned in the introduction, the Bergman fans of matroids satisfy TPD over $\mathbb{R}$ and $\mathbb{Z}$ [JRS18; JSS19], however these are not the only such fans, as can be seen from the next example.

Example I.4.4. Let $f_1 \coloneqq (0,1,1,1)$, $f_2 \coloneqq (1,0,-1,1)$, $f_3 \coloneqq (1,1,0,-1)$ and $f_4 \coloneqq (1,-1,1,0)$ be vectors in $\mathbb{R}^4$ and let e_1, e_2, e_3 and e_4 be the standard basis. Consider the fan generated by the cones of vertices connected by an edge in Figure I.7, so that for instance the cone of e_1 and f_2 is included. This fan was used in [BH17] to construct a counter-example to the strongly positive Hodge conjecture. It is not matroidal, since it does not satisfy the Hard Lefschetz property of [AHK18].

We compute its cellular tropical homology and cohomology over $\mathbb{Q}$ using the cellular sheaves package [KSW17] for [`polymake`], we have:

$$\begin{aligned} H^0(\Sigma, \mathcal{F}^0_{\mathbb{Q}}) &\cong \mathbb{Q} & & & H_2^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}}) &\cong \mathbb{Q} \\ H^0(\Sigma, \mathcal{F}^1_{\mathbb{Q}}) &\cong \mathbb{Q}^4 & &\text{and} & H_2^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Q}}) &\cong \mathbb{Q}^4 \\ H^0(\Sigma, \mathcal{F}^2_{\mathbb{Q}}) &\cong \mathbb{Q}^5 & & & H_2^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Q}}) &\cong \mathbb{Q}^5 \end{aligned}$$

with all other groups being zero. By Proposition I.3.23, the cap product is injective, and since the dimensions agree, the cap products are isomorphisms when they are nonzero. Hence the fan satisfies TPD over $\mathbb{Q}$, where the weights for the fundamental class are chosen so as to form a generator of $H_2^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}}) = \mathbb{Q}$.

I.4.2 Necessary conditions for tropical Poincaré duality

We now turn to giving some necessary conditions for TPD to hold.

First, in light of Proposition I.3.7, the Borel–Moore homology of fans satisfying TPD is concentrated in degree d. Indeed, by Proposition I.3.7, $H^q(\Sigma, \mathcal{F}^p_R) = 0$ for $q \neq 0$, hence the isomorphism $\frown [\Sigma, w]\colon H^q(\Sigma, \mathcal{F}^p_R) \cong H^{BM}_{d-q}(\Sigma, \mathcal{F}^R_{d-p})$ gives $H^{BM}_q(\Sigma, \mathcal{F}^R_{d-p}) = 0$ for $q \neq d$.

Note also that, for (Σ, w) be an R-balanced fan satisfying TPD over R, Σ must be uniquely R-balanced by w. This is because the cap product maps $1 \in R \cong H^0(\Sigma, \mathcal{F}^0_R)$ (see Proposition I.3.7) to $1 \frown [\Sigma, w] = [\Sigma, w] \in H^{BM}_d(\Sigma, \mathcal{F}^R_d)$, which must be a generator. Then by Definition I.3.12, the fan Σ is uniquely R-balanced.

Example I.4.5. For any ring R, the fan in Figure I.1 is R-balanced, but not uniquely R-balanced by Example I.3.8, hence it cannot satisfy TPD over R.

Now, assuming that we are working over a field $\Bbbk$, and that the Borel–Moore homology of the fan vanishes in an appropriate way, we can determine that the fan satisfies TPD through an Euler characteristic argument.

Proposition I.4.6. *Let $\Bbbk$ be a field, and (Σ, w) be a $\Bbbk$-balanced fan of dimension d. Suppose $H_q^{BM}(\Sigma, \mathcal{F}_p^{\Bbbk}) = 0$ for $q \neq d$. Then, for a given p, the cap product*

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}^p_{\Bbbk}) \to H^{BM}_{d-q}(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})$$

is an isomorphism for all q if and only if

$$(-1)^d \chi(C^{BM}_\bullet(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})) = \dim_k \mathcal{F}^p_{\Bbbk}(v). \tag{I.1}$$

Moreover, (Σ, w) satisfies TPD over $\Bbbk$ if and only if Equation (I.1) holds for all p.

Proof. Since the only compact cell in Σ is the vertex v, we have

$$H^q(\Sigma, \mathcal{F}^p_{\Bbbk}) = \begin{cases} \mathcal{F}^p_{\Bbbk}(v) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the vanishing condition on tropical Borel–Moore homology, the cap product $\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}^p_{\Bbbk}) \to H^{BM}_{d-q}(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})$ is then immediately an isomorphism for $q \neq 0$, and

$$\chi(C^{BM}_\bullet(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})) = \chi(H^{BM}_\bullet(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})) = (-1)^d \dim_{\Bbbk} H^{BM}_d(\Sigma, \mathcal{F}^{\Bbbk}_{d-p}). \tag{I.2}$$

Since the cap product is injective by Proposition I.3.23 and we are working over a field, the maps

$$\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}^p_{\Bbbk}) \to H^{BM}_d(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})$$

are isomorphisms if and only if $\dim_{\Bbbk} H^0(\Sigma, \mathcal{F}^p_{\Bbbk}) = \dim_{\Bbbk} \mathcal{F}^p_{\Bbbk}(v)$ is equal to $\dim_{\Bbbk} H^{BM}_d(\Sigma, \mathcal{F}^{\Bbbk}_{d-p})$. By Equation (I.2), this is exactly the claimed result. ∎

I.4.3 Dimension one and codimension one

We completely classify rational polyhedral fans of dimension 1 satisfying TPD over an arbitrary commutative ring. We begin with a utility lemma:

Lemma I.4.7. *Let R be a commutative ring, and (Σ, w) an R-balanced fan of dimension one. Then we have $H^{BM}_0(\Sigma, \mathcal{F}^R_0) = 0$ and $H^{BM}_0(\Sigma, \mathcal{F}^R_1) = 0$.*

Proof. Let $v \in \Sigma$ be the vertex of the fan. By Definition I.3.6, the tropical Borel–Moore cochain complexes are:

$$C^{BM}_\bullet(\Sigma, \mathcal{F}^R_0) \colon \qquad \bigoplus_{\epsilon \in \Sigma^1} R \xrightarrow{\partial^0_1} R \to 0, \quad \text{and}$$

$$C_\bullet^{BM}(\Sigma, \mathcal{F}_1^R)\colon \qquad \bigoplus_{\epsilon\in\Sigma^1} \mathcal{F}_1^R(\epsilon) \xrightarrow{\partial_1^1} \mathcal{F}_1^R(v) \to 0.$$

Here ∂_1^0 is the map given by the matrix $(1\ 1\ \dots\ 1)$. It is surjective and thus $H_0^{BM}(\Sigma, \mathcal{F}_0^R) = 0$. Similarly, $\mathcal{F}_1^R(v) = \sum_{\epsilon\in\Sigma^1} L_R(\epsilon) = \sum_{\epsilon\in\Sigma^1} \mathcal{F}_1^R(\epsilon)$, thus ∂_1^1 is surjective, hence $H_0^{BM}(\Sigma, \mathcal{F}_1^R) = 0$. ■

Theorem I.4.8. *Let R be a commutative ring, and (Σ, w) an R-balanced fan of dimension one. Then (Σ, w) satisfies tropical Poincaré duality over R if and only if it is uniquely R-balanced and all the weights are units in R.*

Proof. We need to show that all four of the following cap products

1. $\frown [\Sigma, w]\colon H^1(\Sigma, \mathcal{F}_R^1) \to H_0^{BM}(\Sigma, \mathcal{F}_0^R)$,
2. $\frown [\Sigma, w]\colon H^1(\Sigma, \mathcal{F}_R^0) \to H_0^{BM}(\Sigma, \mathcal{F}_1^R)$,
3. $\frown [\Sigma, w]\colon H^0(\Sigma, \mathcal{F}_R^0) \to H_1^{BM}(\Sigma, \mathcal{F}_1^R)$,
4. $\frown [\Sigma, w]\colon H^0(\Sigma, \mathcal{F}_R^1) \to H_1^{BM}(\Sigma, \mathcal{F}_0^R)$,

are isomorphisms if and only if (Σ, w) is uniquely R-balanced and all the weights are units in R. We will show this in three parts:

(a) First, we show that the maps (1) and (2) are trivial maps between zero-modules.

(b) Then we show that (3) being an isomorphism is the definition of being uniquely R-balanced.

(c) Finally, we show that (4) is an isomorphism if and only if (Σ, w) is uniquely R-balanced, with the added condition that all the weights are units in R.

In total, this will then show that (Σ, w) satisfies tropical Poincaré duality over R if and only if it is uniquely R-balanced and all the weights are units in R.

Beginning with (a), by Lemma I.4.7 and Proposition I.3.7, all involved modules are zero. Moreover the cap product map is zero by Proposition I.3.22, hence the maps (1) and (2) are trivially isomorphisms.

Next for (b), the map $\frown [\Sigma, w]\colon H^0(\Sigma, \mathcal{F}_R^0) \to H_1^{BM}(\Sigma, \mathcal{F}_1^R)$ is given by sending a scalar $\alpha \in H^0(\Sigma, \mathcal{F}_R^0) \cong R$ to $\alpha \frown [\Sigma, w]$. The 0-contraction of a scalar is multiplication by this scalar, so that $\alpha \frown [\Sigma, w] = \alpha \cdot [\Sigma, w]$. It is therefore an isomorphism if and only if $\langle [\Sigma, w]\rangle$ generates $H_1^{BM}(\Sigma, \mathcal{F}_1^R)$, which is the definition of uniquely R-balanced (Definition I.3.12).

Finally, we turn to (c). We begin with some notation. Let v be the vertex of Σ and number the one-dimensional rays as $\epsilon_1, \dots, \epsilon_m$, with weights $w_i = w(\epsilon_i)$. The Borel–Moore cochain group is $C_1^{BM}(\Sigma, \mathcal{F}_0^R) = \oplus_{i=1}^m R$, which has a basis $x_1, \dots, x_m$, with x_i corresponding to ϵ_i. The elements $x_i - x_m \in C_1^{BM}(\Sigma, \mathcal{F}_0^R)$, for $i = 1, \dots, m-1$ form a basis for $H_1^{BM}(\Sigma, \mathcal{F}_0^R) = \ker(1\ 1\ \dots\ 1)$. For each ϵ_i, we select the generator $\Lambda_i \in L_R(\epsilon_i)$ compatible with the orientation of ϵ_i, and let $\Theta_i := \iota_{\epsilon_i,v}(\Lambda_i)$ be its image under the inclusion $\iota_{\epsilon_i,v}\colon \mathcal{F}_1^R(\epsilon_i) \to \mathcal{F}_1^R(v)$. Thus the fundamental class $[\Sigma, w] \in H_1^{BM}(\Sigma, \mathcal{F}_R^1)$ is explicitly the element $(w_i\Lambda_i)_{i=1}^m \in H_1^{BM}(\Sigma, \mathcal{F}_1^R) = \ker((\iota_{\epsilon_i,v})_{i=1}^m) \subset C_1^{BM}(\Sigma, \mathcal{F}_1^R)$. The cap product map $\frown [\Sigma, w]\colon H^0(\Sigma, \mathcal{F}_R^1) \to H_1^{BM}(\Sigma, \mathcal{F}_0^R)$ takes a covector $\phi \in H^0(\Sigma, \mathcal{F}_R^1) = \mathcal{F}_R^1(v)$ to the element

$$(w_i\phi(\Theta_i))_{i=1}^m \in H_1^{BM}(\Sigma, \mathcal{F}_0^R).$$

Now, suppose all the weights w_i are units in R and (Σ, w) is uniquely R-balanced. Then the elements $w_i\Theta_i$, for $i = 1, \ldots, m-1$, form a basis for $\mathcal{F}_1^R(v)$, with the corresponding dual basis $w_i^{-1}\Theta_i^*$ for $\mathcal{F}_R^1(v)$. Then, for each $j = 1, \ldots, m-1$,

$$w_j^{-1}\Theta_j^* \frown [\Sigma, w] = (w_i w_j^{-1}\Theta_j^*(\Theta_i))_{i=1}^m = (0, \ldots, 0, 1, 0, \ldots, w_m w_j^{-1}\Theta_j^*(\Theta_m)),$$

where the only two non-zero entries are in the j-th and m-th positions. Since this is a cycle in $C_1^{BM}(\Sigma, \mathcal{F}_0^R)$, we must have $1 + w_m w_j^{-1}\Theta_j^*(\Theta_m) = 0$, so that

$$w_j^{-1}\Theta_j^* \frown [\Sigma, w] = x_i - x_m.$$

Thus the images of the basis elements $w_j^{-1}\Theta_j^*$ of $\mathcal{F}_1^R(v)$ form a basis of $H_1^{BM}(\Sigma, \mathcal{F}_0^R)$, hence $\frown [\Sigma, w]$ is an isomorphism.

For the converse direction, we show that if either the weights are non-units or the fan is not uniquely R-balanced, then the cap product is not an isomorphism.

First, suppose some weight w_k is not a unit in R. Then for any $\phi \in H^0(\Sigma, \mathcal{F}_R^1) = \mathcal{F}_R^1(v)$, the k-th component of $\phi \frown [\Sigma, w]$ is contained in the ideal $\langle w_k \rangle \subset R$, which does not contain 1. Hence the element $x_k - x_m$ of $H_1^{BM}(\Sigma, \mathcal{F}_0^R)$ cannot be in the image of $\frown [\Sigma, w]$, which is therefore not surjective and hence not an isomorphism.

Finally, suppose that Σ is not uniquely R-balanced. Since $H_1^{BM}(\Sigma, \mathcal{F}_0^R)$ is free of rank $m-1$, we may assume that $\mathcal{F}_R^1(v)$ is as well, otherwise there cannot be an isomorphism. Since Σ is not uniquely R-balanced, $\operatorname{rank}_R H_1^{BM}(\Sigma, \mathcal{F}_1^R) > 1$, so that by working with the Euler characteristics, we must have $\operatorname{rank}_R \mathcal{F}_1^R(v) < m-1$. Dualizing, we obtain that $\operatorname{rank}_R \mathcal{F}_R^1(v) < m-1 = \operatorname{rank}_R H_1^{BM}(\Sigma, \mathcal{F}_0^R)$ and so the cap product cannot be an isomorphism. ■

Corollary I.4.9. *Let $\Bbbk$ be a field, (Σ, w) a $\Bbbk$-balanced fan of dimension one. Then (Σ, w) satisfies TPD over $\Bbbk$ if and only if it is uniquely $\Bbbk$-balanced.*

Proof. By Theorem I.4.8, (Σ, w) satisfies TPD if and only if it is uniquely $\Bbbk$-balanced, and all the weights are units in $\Bbbk$. The weights are non-zero by Definition I.3.11, hence must be units since $\Bbbk$ is a field. ■

Example I.4.10. Let $\Sigma \subset \mathbb{Z}^3$ be the 1-dimensional fan with a vertex at the origin, and the four cones $\sigma_1, \sigma_2, \sigma_3$ and σ_4 generated by the vectors $\nu_1 = (1, 0, 2)$, $\nu_2 = (-1, 0, 0)$, $\nu_3 = (0, -1, 0)$, $\nu_4 = (0, 1, -2)$ respectively. This is a balanced fan with the constant unit weight function $w(\sigma_i) = 1$. The Borel–Moore chain complex $C_\bullet^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}})$ can be written as

$$0 \longrightarrow \langle\Lambda_1\rangle_{\mathbb{Z}} \oplus \langle\Lambda_2\rangle_{\mathbb{Z}} \oplus \langle\Lambda_3\rangle_{\mathbb{Z}} \oplus \langle\Lambda_4\rangle_{\mathbb{Z}} \xrightarrow{(\iota_{\sigma_i,v})} \mathcal{F}_1^{\mathbb{Z}}(v) \longrightarrow 0$$

Since $\iota_{\sigma_i,v}(\Lambda_i) = \nu_i$, we see in fact that $H_1^{BM}(\Sigma, \mathcal{F}_1^{\mathbb{Z}}) = \langle[\Sigma, w]\rangle$, where $[\Sigma, w] = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$. Thus (Σ, w) is also uniquely $\mathbb{Z}$-balanced. Moreover, the complex $C_\bullet^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}})$ is

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{(1\,1\,1\,1)} \mathbb{Z} \longrightarrow 0,$$

so that $H_1^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) \cong \mathbb{Z}^3$. We pick the basis ν_1, ν_2, ν_3 for $\mathcal{F}^1_{\mathbb{Z}}(v)$, and balancing gives $\nu_4 = -\nu_1 - \nu_2 - \nu_3$. The dual basis for $\mathcal{F}^1_{\mathbb{Z}}(v)$ is $\nu_1^*, \nu_2^*, \nu_3^*$. We see that

$$\begin{aligned}
\nu_1^* \frown [\Sigma, w] &= (\oplus \lrcorner\, \Lambda_{\sigma_i})(\oplus \rho_{v,\sigma_i})(\nu_1^*) \\
&= (\oplus \lrcorner\, \Lambda_{\sigma_i})(\oplus(\nu_1^* \circ \iota_{\sigma_i,v})) \\
&= (\nu_1^*(\iota_{\sigma_1,v}(\Lambda_1)),\ \nu_1^*(\iota_{\sigma_2,v}(\Lambda_2)),\ \nu_1^*(\iota_{\sigma_3,v}(\Lambda_3)),\ \nu_1^*(\iota_{\sigma_4,v}(\Lambda_4))) \\
&= (\nu_1^*(\nu_1), \nu_1^*(\nu_2), \nu_1^*(\nu_3), \nu_1^*(\nu_4)) \\
&= (1, 0, 0, -1).
\end{aligned}$$

Similarly, $\nu_2^* \frown [\Sigma, w] = (0, 1, 0, -1)$, and $\nu_3^* \frown [\Sigma, w] = (0, 0, 1, -1)$. Since the images of the generating set y_1^*, y_2^*, y_3^* for $\mathcal{F}^1_{\mathbb{Z}}(v)$ is a generating set for $H_1^{BM}(\Sigma, \mathcal{F}_0^{\mathbb{Z}}) \cong \mathbb{Z}^3$, the cap product is an isomorphism.

For codimension 1 fan tropical cycles in $\mathbb{R}^n$, which are fan tropical hypersurfaces, we can characterize the Newton polytopes of the hypersurfaces having TPD. We refer to [MR18, Chap. 2] for background on tropical hypersurfaces in $\mathbb{R}^n$, which they call *very affine tropical hypersurfaces.*

Proposition I.4.11. *Let $f \in \mathbb{T}[x_0^{\pm 1}, \ldots, x_d^{\pm 1}]$ be a tropical Laurent polynomial such that the very affine tropical cycle $X = V(f) \subset \mathbb{R}^{d+1}$ is supported on a pointed fan. If X satisfies TPD over a commutative ring R, then the Newton polytope $\Delta(f)$ of f is a simplex.*

Proof. By assumption, the very affine tropical cycle X is a pointed d-dimensional rational polyhedral fan ([MR18, Cor 2.3.2]), thus $H^q(X, \mathcal{F}_R^p) = 0$ for all $q > 0$ and all p, and the isomorphisms $\frown [X]\colon H^q(X, \mathcal{F}_R^p) \to H_{d-q}^{BM}(X, \mathcal{F}_{d-p}^R)$ give in particular that $H_{d-q}^{BM}(X, \mathcal{F}_0^R) = 0$ for all $q > 0$.

Since X is the d-skeleton of the dual fan to $\Delta(f)$ by [MR18, Thm 2.3.7, Cor 2.3.2], $\dim_R H_d^{BM}(X, \mathcal{F}_0)$ is $\#\mathrm{Vert}(\Delta(f)) - 1$, the number of vertices of the polytope $\Delta(f)$, minus 1.

Since X is d-dimensional, $\dim_R H^0(X, \mathcal{F}_R^p) = \dim_R \mathcal{F}_R^p(v) = \binom{d+1}{p}$, thus by Poincaré duality we have

$$\#\mathrm{Vert}(\Delta(f)) - 1 = \dim_R H_d^{BM}(X, \mathcal{F}_0^R) = \dim_R H^0(X, \mathcal{F}_R^d) = \binom{d+1}{d} = d+1,$$

and so $\Delta(f)$, being $(d+1)$-dimensional and having $d+2$ vertices, is a simplex. ■

I.5 Local tropical Poincaré duality spaces

In this section, we study I.1.2. In Section I.5.1, we prove Theorem I.5.4. This theorem implies that TPD on faces of a fan, along with vanishing of its tropical BM homology, gives TPD on the whole fan. A version of the proof gives a partial classification of TPD spaces of dimension two. Using Theorem I.5.4, we prove Theorem I.5.10 in Section I.5.2, which states that local TPD spaces are exactly fans whose codimension one faces are local TPD spaces, and all of whose faces have vanishing tropical BM homology. Finally, we use the dimension one classification from Theorem I.4.8 to give a more geometric characterization of local TPD spaces in Corollary I.5.11.

I.5.1 TPD from faces

We fix a principal ideal domain R, and we use the following shortened notation $H^{BM}_{d,d-p}(\Sigma; R) \coloneqq H^{BM}_d(\Sigma, \mathcal{F}^R_{d-p})$. We prove Theorem I.5.4 in two steps: The first step will be to show Proposition I.5.2, which relates the cellular chain complex $C^\bullet_c(\Sigma, \mathcal{F}^p_R)$ to a complex involving the Borel–Moore homology groups $H^{BM}_d(\gamma_\succeq, \mathcal{F}^R_{d-p})$ for faces $\gamma \in \Sigma$ by using the cap product, which we show is exact. We then prove the theorem by showing that TPD on the faces, along with exactness in the mentioned complex, imply Poincaré duality for the whole fan.

Let Σ be a d-dimensional rational polyhedral fan. For each maximal face $\alpha \in \Sigma^d$, the constant sheaf $\mathcal{F}^R_{d-p}(\alpha)_{\alpha_\preceq}$ gives a cochain complex $(C^\bullet_c(\alpha_\preceq, \mathcal{F}^R_{d-p}(\alpha)_{\alpha_\preceq}), d^\bullet_\alpha)$. Taking the direct sum of these for all $\alpha \in \Sigma^d$, we obtain a complex

$$(A^\bullet, d^\bullet) \coloneqq (\oplus_\alpha C^\bullet_c(\alpha_\preceq, \mathcal{F}^R_{d-p}(\alpha)_{\alpha_\preceq}), \oplus_\alpha d^\bullet_\alpha). \tag{I.3}$$

The i-th term of this complex is given by

$$A^i = \oplus_\alpha C^i_c(\alpha_\preceq, \mathcal{F}^R_{d-p}(\alpha)_{\alpha_\preceq}) = \oplus_{\alpha \in \Sigma^d} \oplus_{\substack{\gamma \in \Sigma^i \\ \gamma \prec \alpha}} \mathcal{F}^R_{d-p}(\alpha).$$

Rearranging terms, we may use Proposition I.3.10 to obtain an inclusion

$$\oplus_{\gamma \in \Sigma^i} H^{BM}_{d,d-p}(\gamma_\succeq; R) \subseteq A^i.$$

Proposition I.5.1. *There is a cochain complex* $(\oplus_{\gamma \in \Sigma^\bullet} H^{BM}_{d,d-p}(\gamma_\succeq; R), \overline{d^\bullet})$, *which is the restriction of the cochain complex* $(A^\bullet, d^\bullet)$ *from Equation* (I.3).

Proof. It suffices to show that, for each $i \geq 0$,

$$d^i\left(\oplus_{\gamma \in \Sigma^i} H^{BM}_{d,d-p}(\gamma_\succeq; R)\right) \subseteq \oplus_{\kappa \in \Sigma^{i+1}} H^{BM}_{d,d-p}(\kappa_\succeq; R).$$

This follows from a direct computation. ∎

Proposition I.5.2. *For (Σ, w) an R-balanced fan of dimension $d \geq 2$, there is a commutative diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{F}_R^p(v) & \xrightarrow{\delta^0} & \bigoplus_{\tau\in\Sigma^1} \mathcal{F}_R^p(\tau) & \xrightarrow{\delta^1} & \bigoplus_{\sigma\in\Sigma^2} \mathcal{F}_R^p(\sigma) & \xrightarrow{\delta^2} & \cdots \\ \downarrow{\frown[\Sigma,w]} & & \downarrow{\oplus_\tau \frown[\tau_{\succeq},w]} & & \downarrow{\oplus_\sigma \frown[\sigma_{\succeq},w]} & & \\ 0 \to H^{BM}_{d,d-p}(\Sigma;R) & \xrightarrow{\oplus_\alpha \overline{d^0_\alpha}} & \bigoplus_{\tau\in\Sigma^1} H^{BM}_{d,d-p}(\tau_{\succeq};R) & \xrightarrow{\oplus_\alpha \overline{d^1_\alpha}} & \bigoplus_{\sigma\in\Sigma^2} H^{BM}_{d,d-p}(\sigma_{\succeq};R) & \xrightarrow{\oplus_\alpha \overline{d^2_\alpha}} & \cdots \end{array} \tag{I.4}$$

with all the vertical maps being injective, where the upper row is given by the complex $(C_c^\bullet(\Sigma, \mathcal{F}_p^R), \delta^\bullet)$, and the lower row is the complex $(\oplus_{\gamma\in\Sigma^\bullet} H^{BM}_{d,d-p}(\gamma_{\succeq}; R), \overline{d^\bullet})$ from Proposition I.5.1.

Proof. First, we wish to show that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{F}_R^p(v) & \xrightarrow{\delta^0} & \bigoplus_{\tau\in\Sigma^1} \mathcal{F}_R^p(\tau) & \xrightarrow{\delta^1} & \bigoplus_{\sigma\in\Sigma^2} \mathcal{F}_R^p(\sigma) & \xrightarrow{\delta^2} & \cdots \\ \downarrow{\frown \mathrm{Ch}(\Sigma,w)} & & \downarrow{\oplus_\tau \frown \mathrm{Ch}(\tau_{\succeq},w)} & & \downarrow{\oplus_\sigma \frown \mathrm{Ch}(\sigma_{\succeq},w)} & & \\ 0 \to \bigoplus_{\alpha\in\Sigma^d} \mathcal{F}^R_{d-p}(\alpha) & \xrightarrow{\oplus_\alpha d^0_\alpha} & \bigoplus_{\tau\in\Sigma^1} \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\tau}} \mathcal{F}^R_{d-p}(\alpha) & \xrightarrow{\oplus_\alpha d^1_\alpha} & \bigoplus_{\sigma\in\Sigma^2} \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\sigma}} \mathcal{F}^R_{d-p}(\alpha) & \xrightarrow{\oplus_\alpha d^2_\alpha} & \cdots. \end{array}$$

The upper row is the compact support complex $(C_c^\bullet(\Sigma, \mathcal{F}_R^p), \delta^\bullet)$ for the $\mathcal{F}_R^p$ cohomology (see Definition I.2.13). The lower row is the complex $(A^\bullet, d^\bullet)$ from (I.3), where the order of indexing is changed for clarity in relation to the cap morphism.

The first vertical map in diagram (I.4) is given by the cap product on the chain level of (Σ, w), as in Definition I.3.19. For the r-th column, the vertical map is given as the direct sum over all $\gamma \in \Sigma^r$ of the maps:

$$\begin{aligned} \frown \mathrm{Ch}(\gamma_{\succeq}, w) \colon \mathcal{F}_R^p(\gamma) &\to \oplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\gamma}} \mathcal{F}^R_{d-p}(\alpha) \\ v &\mapsto (v_\gamma \lrcorner\, w(\alpha)\Lambda_\alpha)_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\gamma}}. \end{aligned}$$

To obtain commutativity of the described diagram, we select one square and show commutativity there:

$$\begin{array}{ccc} \bigoplus_{\gamma\in\Sigma^r} \mathcal{F}_R^p(\gamma) & \xrightarrow{\delta^r} & \bigoplus_{\kappa\in\Sigma^{r+1}} \mathcal{F}_R^p(\kappa) \\ \downarrow{\oplus_\gamma \frown \mathrm{Ch}(\gamma_{\succeq},w)} & & \downarrow{\oplus_\kappa \frown \mathrm{Ch}(\kappa_{\succeq},w)} \\ \bigoplus_{\gamma\in\Sigma^r} \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\gamma}} \mathcal{F}^R_{d-p}(\alpha) & \xrightarrow{\oplus_\alpha d^r_\alpha} & \bigoplus_{\kappa\in\Sigma^{r+1}} \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\kappa}} \mathcal{F}^R_{d-p}(\alpha) \end{array} \tag{I.5}$$

For $v = (v_\gamma)_{\gamma \in \Sigma^r} \in \bigoplus_{\gamma \in \Sigma^r} \mathcal{F}^p_R(\gamma)$, we can expand the definitions for the right then down composition to get

$$\begin{aligned}((\oplus_\kappa \frown \operatorname{Ch}(\kappa_{\succeq}, w) \circ \delta^r)(v) &= (\oplus_\kappa \frown \operatorname{Ch}(\kappa_{\succeq}, w)) \left(\left(\textstyle\sum_{\gamma \prec \kappa} \mathcal{O}(\gamma,\kappa) v_\gamma\right)_{\kappa \in \Sigma^{r+1}} \right) \\ &= \left(\left(\textstyle\sum_{\gamma \prec \kappa} \mathcal{O}(\gamma,\kappa) v_\gamma\right) \lrcorner\, w(\alpha)\Lambda_\alpha \right)_{\substack{\kappa \in \Sigma^{r+1} \\ \alpha \in \Sigma^d \\ \alpha \succ \kappa}} .\end{aligned}$$

For the down then right composition, we get

$$\begin{aligned}((\oplus_\alpha d^r_\alpha) \circ (\oplus_\gamma \frown \operatorname{Ch}(\gamma_{\succeq}, w)))(v) &= (\oplus_\alpha d^r_\alpha) \left((w(\alpha) v_\gamma \lrcorner\, \Lambda_\alpha)_{\substack{\gamma \in \Sigma^{r+1},\, \alpha \in \Sigma^d \\ \alpha \succ \gamma}} \right) \\ &= \left(\textstyle\sum_{\gamma \prec \kappa} \mathcal{O}(\gamma,\kappa)(v_\gamma \lrcorner\, w(\alpha)\Lambda_\alpha) \right)_{\substack{\kappa \in \Sigma^{r+1} \\ \alpha \in \Sigma^d \\ \alpha \succ \kappa}} .\end{aligned}$$

Comparing the two above equations, diagram (I.5) is commutative since the contraction $\lrcorner\, w(\alpha)\Lambda_\alpha$ is R-linear.

Lastly, we we wish to show injectivity of the vertical maps, when restricting to the Borel–Moore homology groups. By Proposition I.3.24 and Proposition I.3.10, for each $\kappa \in \Sigma$, we have that

$$H^{BM}_d(\kappa_{\succeq}, \mathcal{F}^R_{d-p}) \cong \ker \left(\oplus_{\substack{\alpha \in \Sigma^d \\ \alpha \succ \kappa}} \mathcal{F}^R_{d-p}(\alpha) \to \oplus_{\substack{\beta \in \Sigma^{d-1} \\ \beta \succ \kappa}} \mathcal{F}^R_{d-p}(\beta) \right),$$

and the given formulas for the maps $\oplus_{\kappa \in \Sigma^{r+1}} \frown \operatorname{Ch}(\kappa_{\succeq}, w)$ correspond exactly to the cap products in homology

$$\oplus_\kappa \frown [\kappa_{\succeq}, w] \colon \bigoplus_{\kappa \in \Sigma^{r+1}} \mathcal{F}^p_R(\kappa) \to \bigoplus_{\kappa \in \Sigma^{r+1}} H^{BM}_d(\kappa_{\succeq}, \mathcal{F}^R_{d-p}).$$

We have the following diagram when only considering the images

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{F}^p_R(v) & \xrightarrow{\delta^0} & \bigoplus_{\tau \in \Sigma^1} \mathcal{F}^p_R(\tau) & \xrightarrow{\delta^1} & \bigoplus_{\sigma \in \Sigma^2} \mathcal{F}^p_R(\sigma) & \xrightarrow{\delta^2} & \cdots \\
 & \Big\downarrow {\scriptstyle \frown[\Sigma,w]} & & \Big\downarrow {\scriptstyle \oplus_\tau \frown [\tau_{\succeq}, w]} & & \Big\downarrow {\scriptstyle \oplus_\sigma \frown [\sigma_{\succeq}, w]} & & \\
0 \longrightarrow & H^{BM}_{d,d-p}(\Sigma;R) & \xrightarrow{\oplus_\alpha \overline{d^0_\alpha}} & \bigoplus_{\tau \in \Sigma^1} H^{BM}_{d,d-p}(\tau_{\succeq};R) & \xrightarrow{\oplus_\alpha \overline{d^1_\alpha}} & \bigoplus_{\sigma \in \Sigma^2} H^{BM}_{d,d-p}(\sigma_{\succeq};R) & \xrightarrow{\oplus_\alpha \overline{d^2_\alpha}} & \cdots
\end{array} \tag{I.6}$$

where the cochain differentials in the lower row have been restricted.

These vertical maps are the direct sum of cap products, so by Proposition I.3.23, are injective. ■

Proposition I.5.3. *Let Σ be a fan of dimension $d \geq 2$ such that $H^{BM}_q(\gamma_{\succeq}, \mathcal{F}^R_p) = 0$, for $q \neq d$ and all p, for each face $\gamma \in \Sigma$. Then the complex $(\oplus_{\gamma \in \Sigma^\bullet} H^{BM}_{d,d-p}(\gamma_{\succeq};R), \overline{d^\bullet})$ in Proposition I.5.1 is exact except in the rightmost position.*

Proof. We will now construct a double complex $K_{\bullet,\bullet}$, which corresponds to the Cartan-Eilenberg resolution of diagram (I.4), using that these are the homology groups of the complexes $C^{BM}_\bullet(\gamma_{\succeq}, \mathcal{F}^R_{d-p})$. Then, we will use the two spectral sequences converging to the homology of the total complex $H_\bullet(\mathrm{Tot}(K_{\bullet,\bullet}))$ to deduce that the complex is exact except in the rightmost position.

Let $(K_{\bullet,\bullet}, d^0_\wedge, d^0_>)$ be the first-quadrant double complex given by

$$K_{r,s} = \bigoplus_{\kappa \in \Sigma^r} \bigoplus_{\substack{\gamma \in \Sigma^{d-s} \\ \gamma \succeq \kappa}} \mathcal{F}^R_{d-p}(\gamma),$$

for $r \geq 0$, $s \geq 0$. Since all the indices used are relating to the dimensions of particular faces of the fan Σ, this is a first-quadrant double complex.

The vertical differential $(d^0_\wedge)_{r,s} \colon K_{r,s} \to K_{r,s+1}$ is the direct sum over the differentials $\partial^\kappa_\bullet$ of the chain complex for tropical Borel–Moore homology on the star $\kappa_{\succeq}$ for each face $\kappa \in \Sigma^r$, i.e. $(d^0_\wedge)_{r,s} = \oplus_{\kappa \in \Sigma^r} \partial^\kappa_{d-s}$ from Proposition I.3.10.

The horizontal differential $(d^0_>)_{r,s} \colon K_{r,s} \to K_{r+1,s}$ is the direct sum over the differentials $d^\bullet_\gamma$ in the complex of cochains of compact support for the constant sheaf taking value $\mathcal{F}^R_{d-p}(\gamma)$ on the cone $\gamma_{\preceq}$, truncated in degree 1, for each face $\gamma \in \Sigma^{d-s}$. Explicitly, $(d^0_>)_{r,s} = \oplus_{\gamma \in \Sigma^{d-s}} d^r_\gamma$, where $d^\bullet_\gamma$ comes from the complex:

$$0 \to \mathcal{F}^R_{d-p}(\gamma) \xrightarrow{d^0_\gamma} \bigoplus_{\substack{\tau \in \Sigma^1 \\ \gamma \succ \tau}} \mathcal{F}^R_{d-p}(\gamma) \xrightarrow{d^1_\gamma} \dots \xrightarrow{d^{s-2}_\gamma} \bigoplus_{\substack{\kappa \in \Sigma^{s-1} \\ \gamma \succ \kappa}} \mathcal{F}^R_{d-p}(\gamma) \xrightarrow{d^{s-1}_\gamma} \mathcal{F}_{d-p}(\gamma) \to 0,$$

from Proposition I.3.10.

We have that $d^0_\wedge \circ d^0_\wedge = 0$ and $d^0_> \circ d^0_> = 0$ since both are direct sums of differentials of complexes. Moreover, we have $d^0_> \circ d^0_\wedge = d^0_\wedge \circ d^0_>$ which can be checked directly.

Now, since $K_{\bullet,\bullet}$ is a double complex, we can look at the two associated spectral sequences converging to the homology of the total complex $(\mathrm{Tot}(K_{\bullet,\bullet}), d_{\mathrm{T}})$ given by $\mathrm{Tot}(K_{\bullet,\bullet})_m = \prod_{r+s=m} K_{r,s}$ with differential $d_{\mathrm{T}} = d_> + d_\wedge$. We refer to [Wei94, Chapter 5.6] for details.

First, we take the spectral sequence E^r, with $E^0 = K_{\bullet,\bullet}$ and the first differential d^0 being the horizontal differential $d^0_>$ of $K_{\bullet,\bullet}$, which is equivalent to computing the homology row by row. Since the rows $K_{\bullet,s}$ are the complexes $\bigoplus_{\gamma \in \Sigma^{d-s}} C^\bullet_c(\gamma_{\preceq}, F(\gamma)_{\gamma_{\preceq}})$ with $F(\gamma) \coloneqq \mathcal{F}^R_{d-p}(\gamma)$, observing that this is merely the reduced $F(\gamma)$-cohomology of a polytope over which γ is a cone, gives

$$H_k(K_{\bullet,s}, d^0_>) \cong 0$$

for each $s \neq d$. In degree d, we have $H_k(K_{\bullet,d}, d^0_>) \cong \mathcal{F}^R_{d-p}(v)$ for $k = 0$ and $H_k(K_{\bullet,d}, d^0_>) = 0$ otherwise. Thus, the E^1 page becomes

$$E^1_{r,s} \cong \begin{cases} \mathcal{F}^R_{d-p}(v) & r = 0 \text{ and } s = d, \\ 0 & \text{otherwise.} \end{cases}$$

There are now no further non-zero differentials of the spectral sequence, so the E^1 page is the E^∞ page. In particular, we conclude that

$$H_q(\mathrm{Tot}(K_{\bullet,\bullet})_\bullet) = \begin{cases} \mathcal{F}^R_{d-p}(v) & \text{for } q = d, \\ 0 & \text{otherwise.} \end{cases} \tag{I.7}$$

Next, we consider the spectral sequence $\overline{E^r}$, with $\overline{E^0} = K_{\bullet,\bullet}$ and the first differential d^0 being the vertical differential $d^0_\wedge$ of $K_{\bullet,\bullet}$. Taking this differential is therefore equivalent to computing the homology column by column. The r-th column is the direct sum over each $\gamma \in \Sigma^r$ of the complex

$$0 \longrightarrow \bigoplus_{\substack{\alpha\in\Sigma^d \\ \alpha\succ\gamma}} \mathcal{F}^R_{d-p}(\alpha) \xrightarrow{\partial^\gamma_n} \bigoplus_{\substack{\beta\in\Sigma^{d-1} \\ \beta\succ\gamma}} \mathcal{F}^R_{d-p}(\beta) \xrightarrow{\partial^\gamma_{n-1}} \dots$$

$$\dots \xrightarrow{\partial^\gamma_{l+2}} \bigoplus_{\substack{\kappa\in\Sigma^{l+1} \\ \kappa\succ\gamma}} \mathcal{F}^R_{d-p}(\kappa) \xrightarrow{\partial^\gamma_{l+1}} \mathcal{F}^R_{d-p}(\gamma) \longrightarrow 0$$

each of which has the tropical Borel–Moore homology of the star $\gamma_\succeq$ by Proposition I.3.10. Since by assumption, $H^{BM}_q(\gamma_\succeq, \mathcal{F}^R_p) = 0$ for $q \neq d$ and all p, for each face $\gamma \in \Sigma$, we find:

$$H_k(K_{r,\bullet}, d^\bullet_\wedge) = \begin{cases} \oplus_{\kappa\in\Sigma^r} H^{BM}_d(\kappa_\succeq, \mathcal{F}^R_{d-p}) & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the $\overline{E^1}$ page has only the bottom row:

$$0 \to H^{BM}_d(\Sigma, \mathcal{F}^R_{d-p}) \xrightarrow{\oplus_\alpha \overline{d^0_\alpha}} \bigoplus_{\tau\in\Sigma^1} H^{BM}_d(\tau_\succeq, \mathcal{F}^R_{d-p}) \xrightarrow{\oplus_\alpha \overline{d^1_\alpha}} \bigoplus_{\sigma\in\Sigma^2} H^{BM}_d(\sigma_\succeq, \mathcal{F}^R_{d-p}) \xrightarrow{\oplus_\alpha \overline{d^2_\alpha}} \dots$$

The $\overline{E^2}$ page is then merely the homology of this complex, and since it is concentrated in one row, this must be the $\overline{E^\infty}$ page. In particular, by (I.7), the complex only has homology in the rightmost position. ∎

Theorem I.5.4. *Let R be a principal ideal domain, and (Σ, w) be an R-balanced fan of dimension $d \geq 2$, with $H^{BM}_q(\Sigma, \mathcal{F}^R_p) = 0$ for $q \neq d$, for all p. If $(\gamma_\succeq, w)$ satisfies TPD over R, for each $\gamma \in \Sigma$ with $\gamma_\succeq \neq \Sigma$, then (Σ, w) satisfies TPD over R.*

Proof. By assumption $H^{BM}_q(\Sigma, \mathcal{F}^R_{d-p}) = 0$ for $q \neq d$, and $H^q(\Sigma, \mathcal{F}^p_R) = 0$ for $q \neq 0$ by Remark I.2.15, for all p. Thus the cap product map

$$\frown [\Sigma, w] \colon H^q(\Sigma, \mathcal{F}^p_R) \to H^{BM}_{d-q}(\Sigma, \mathcal{F}^R_{d-p})$$

is an isomorphism for all $q = 1, \dots, d$, for all p, and it remains to show that

$$\frown [\Sigma, w] \colon H^0(\Sigma, \mathcal{F}^p_R) \to H^{BM}_d(\Sigma, \mathcal{F}^R_{d-p})$$

is an isomorphism for all p.

Since $H_q^{BM}(\Sigma, \mathcal{F}_{d-p}^R) = 0$ for $q \neq d$ and $(\gamma_{\succeq}, w)$ satisfies TPD over R for each $\gamma \in \Sigma$ with $\gamma_{\succeq} \neq \Sigma$ gives that $H_{d-q}^{BM}(\gamma_{\succeq}, \mathcal{F}_{d-p}^R) = 0$ for all $\gamma \in \Sigma$. Thus, by Proposition I.5.3, the lower row in diagram (I.4) is exact in all degrees except d.

Moreover, the upper row of diagram (I.4) is the complex $C_c^\bullet(\Sigma, \mathcal{F}_R^p)$, which can be seen to be the dual complex to $C_\bullet^{BM}(\Sigma, \mathcal{F}_p^R)$ by the definitions (Definitions I.2.13, I.2.14 and I.3.1).

The complex $C_\bullet^{BM}(\Sigma, \mathcal{F}_p^R)$ consists only of free R-modules, since $\mathcal{F}_p^R(\gamma)$ is a sublattice of $N \otimes_{\mathbb{Z}} R$ for all $\gamma \in \Sigma$, and the ring R is a principal ideal domain, hence we may apply the Universal Coefficient Theorem for cohomology [Wei94, Theorem 3.6.5]. Thus, for each q, we have:

$$0 \to \operatorname{Ext}_R(H_{q-1}^{BM}(\Sigma, \mathcal{F}_p^R), R) \to H_c^q(\Sigma, \mathcal{F}_p^R) \to \operatorname{Hom}_R(H_{q-1}^{BM}(\Sigma, \mathcal{F}_p^R), R) \to 0.$$

Since we assumed $H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R) = 0$ for $q \neq 0$, for all p, one has $H_c^q(\Sigma, \mathcal{F}_p^R) = 0$ for $q \neq d$, for all p. Hence the upper row of diagram (I.4) is exact except in the last position.

The cokernel complex to the chain complex map in (I.4) gives following short exact sequence of chain complexes:

$$0 \to (C_c^\bullet(\Sigma, \mathcal{F}_p^R), \delta^\bullet) \to \left(\oplus_{\gamma \in \Sigma^\bullet} H_d^{BM}(\gamma_{\succeq}, \mathcal{F}_{d-p}^R),\ \oplus_\alpha \overline{d_\alpha^\bullet}\right) \to \operatorname{coker}(\frown) \to 0.$$

This gives a long exact sequence in homology, and since the two first chain complexes are exact in all but the last position, so is the cokernel chain complex. Thus we have the following exact sequence:

$$0 \to \operatorname{coker}(\frown [\Sigma, w]) \to \operatorname{coker}(\oplus_\tau \frown [\tau_{\succeq}, w]) \to \operatorname{coker}(\oplus_\sigma \frown [\sigma_{\succeq}, w]) \to \cdots$$

Since each of the stars $(\gamma_{\succeq}, w)$ satisfies TPD over R, we have $\operatorname{coker}(\oplus_\gamma \frown [\gamma_{\succeq}, w]) = 0$ and so exactness gives $\operatorname{coker}(\frown [\Sigma, w]) = 0$. Thus $\frown [\Sigma, w]$ is both injective by Proposition I.3.23 and surjective, hence is an isomorphism. ∎

Remark I.5.5. In the proof of Theorem I.5.4, the condition that R is a PID is only used to show that $H_q^{BM}(\Sigma, \mathcal{F}_p^R) = 0$ for all $q \neq d$ implies that $H_c^q(\Sigma, \mathcal{F}_R^p) = 0$ for all $q \neq d$. One can let R be an arbitrary commutative ring if we instead assume this latter condition, giving another variant of the theorem.

It is not sufficient that all the star fans $\gamma_{\succeq}$ of faces $\gamma \in \Sigma$ with $\gamma \neq v$ satisfy TPD. The assumption $H_{d-q}^{BM}(\Sigma, \mathcal{F}_{d-p}^R) = 0$ for $q \neq 0$, for all p, is necessary and not implied by TPD of the faces. This is shown by the following example, which is also studied in [Aks19] and in [AP21, Section 11.1].

Example I.5.6. Let Σ be the fan shown in Figure I.8, where the rays are $e_1, e_2, -e_1, -e_2, -e_1 + e_2 + e_3, -e_1 + e_2 - e_3, e_1 - e_2 + e_3, e_1 - e_2 - e_3$. Each of its two-dimensional faces is maximal, so that the star at these faces is just a two-dimensional linear space, which satisfies TPD. Moreover, each ray has exactly three faces meeting in it, so that the stars are uniquely balanced, and

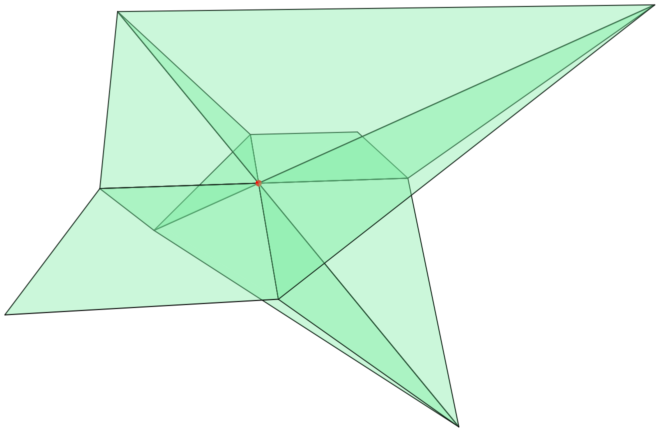

Figure I.8: A fan whose faces excluding the vertex satisfy TPD, but does not itself satisfy it. Figure generated using [`polymake`]

satisfy TPD. We could therefore expect Theorem I.5.4 to give us that the whole fan Σ has TPD.

However, observe that $\dim_{\mathbb{Q}} \mathcal{F}_2^{\mathbb{Q}}(\sigma) = 1$ for each $\sigma \in \Sigma^2$, while $\dim_{\mathbb{Q}} \mathcal{F}_2^{\mathbb{Q}}(\tau) = 2$ for each $\tau \in \Sigma^1$ and $\dim_{\mathbb{Q}} \mathcal{F}_2^{\mathbb{Q}}(v) = 3$. There are 12 two-dimensional faces and 8 one-dimensional faces, so that

$$\chi(C_\bullet^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}})) = 12 - 8 \cdot 2 + 3 = -1.$$

Since $\chi(C_\bullet^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}})) = \chi(H_\bullet^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}}))$, we must have that $H_1^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}}) \ncong 0$. Finally, since $H^1(\Sigma, \mathcal{F}_{\mathbb{Q}}^0) = 0$, the cap product $\frown [\Sigma, w]\colon H^1(\Sigma, \mathcal{F}_{\mathbb{Q}}^0) \to H_1^{BM}(\Sigma, \mathcal{F}_2^{\mathbb{Q}})$ cannot be an isomorphism.

Proposition I.5.7. *Let $\Bbbk$ be a field, and (Σ, w) a $\Bbbk$-balanced fan of dimension 2. Suppose $H_q^{BM}(\Sigma, \mathcal{F}_p^{\Bbbk}) = 0$ for $q \neq 2$, for all p. Then Σ satisfies TPD over $\Bbbk$ if and only if each of the stars $(\tau_\succeq, w)$, for $\tau \in \Sigma^1$ satisfies TPD over $\Bbbk$.*

Proof. First, we show that for each $\sigma \in \Sigma^2$, the star $\sigma_\succeq$ satisfies TPD over $\Bbbk$. For each $\sigma \in \Sigma^2$, we have from Proposition I.3.10 that

$$H^0(\sigma_\succeq, \mathcal{F}_{\Bbbk}^p) = \mathcal{F}_{\Bbbk}^p(\sigma) = \left(\bigwedge^p L_{\mathbb{Z}}(\sigma) \otimes_{\mathbb{Z}} \Bbbk\right)^*,$$
$$H_d^{BM}(\sigma_\succeq, \mathcal{F}_{d-p}^{\Bbbk}) = \mathcal{F}_{d-p}^{\Bbbk}(\sigma) = \bigwedge^{d-p} L_{\mathbb{Z}}(\sigma) \otimes_{\mathbb{Z}} \Bbbk.$$

Moreover, the cap product is an injective map by Proposition I.3.23. These two vector spaces have the same dimension, hence the cap product is an isomorphism.

Next, we consider again the sequence

$$0 \to \operatorname{coker}(\frown [\Sigma, w]) \to \operatorname{coker}(\oplus_\tau \frown [\tau_\succeq, w]) \to \operatorname{coker}(\oplus_\sigma \frown [\sigma_\succeq, w]) \to \cdots$$

from the proof of Theorem I.5.4. Since $\sigma_{\succeq}$ satisfies TPD over k, $\operatorname{coker}(\oplus_\sigma \frown [\sigma_{\succeq}, w]) = 0$, and so $\operatorname{coker}(\frown [\Sigma, w]) \cong \operatorname{coker}(\oplus_\tau \frown [\tau_{\succeq}, w])$. Since both these maps are injective by Proposition I.3.23, the result follows. ■

Remark I.5.8. Theorem I.5.4 shows that under the assumption of the vanishing of Borel–Moore homology, TPD on a fan Σ can be deduced from TPD on its faces. In fact, it is not necessary to assume that all the faces satisfy TPD: In general, the "vertical first" spectral sequence in Proposition I.5.3 degenerates on page 2 when $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$, for $q \neq d$ and all p, for each face $\gamma \in \Sigma$. However, the exactness of the lower row in diagram (I.4) in positions 0 and 1 follow from the weaker assumption that $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$, for $q \neq d$ and all p, for each face $\gamma \in \Sigma^i$, with $i = 0, 1$. One can then show, in a restricted version of the proof of Theorem I.5.4, that TPD for all $\tau \in \Sigma^1$ implies that Σ satisfies TPD.

I.5.2 A characterization of local TPD spaces

We now turn to studying fans which satisfy TPD at every face. Using Theorem I.5.4, we characterize such fans as the ones for which the condition holds in codimension 1 along with a vanishing condition on Borel–Moore homology, which was suggested to us by Amini and Piquerez [AP21].

Definition I.5.9. Let R be a ring, and (Σ, w) an R-balanced fan. If, for each face $\gamma \in \Sigma$, the star fan $\gamma_{\succeq}$ satisfies tropical Poincaré duality over R, we say that Σ is a *local tropical Poincaré duality space over R*.

In particular, this implies that Σ satisfies TPD over R. In the case where $R = \mathbb{Z}$ or $\mathbb{Q}$, being a local TPD space is equivalent to the *tropical smoothness* introduced by Amini and Piquerez in [AP21]. We use a different notion of the star of a face, but the equivalence of the definitions can be seen from [AP21, Proposition 3.17], which in turn follows from the tropical Künneth theorem [GS23, Theorem B].

Theorem I.5.10. *Let R be a principal ideal domain, and (Σ, w) a d-dimensional R-balanced fan. Then Σ is a local TPD space over R if and only if $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for all $\gamma \in \Sigma$ and $q \neq d$, and for all faces β of codimension 1, the star fans $\beta_{\succeq}$ are TPD spaces over R.*

Proof. If Σ is a local TPD space over R, each of the star fans $\gamma_{\succeq}$ for $\gamma \in \Sigma$, in particular the codimension 1 faces are TPD spaces over R. Moreover, this implies that the Borel–Moore homology groups $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R)$ vanish for $q \neq d$ and all $\gamma \in \Sigma$.

Next, assume that the star fans $\beta_{\succeq}$ for $\beta \in \Sigma^{d-1}$ are local TPD spaces over R and $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for all $\gamma \in \Sigma$. First, we apply Theorem I.5.4 to all faces of codimension two μ in Σ. For a given $\mu \in \Sigma^{d-2}$, we have $H_q^{BM}(\mu_{\succeq}, \mathcal{F}_p^R) = 0$ for all p by assumption. Moreover, each face $\widetilde{\beta} \in \mu_{\succeq}$ is a subdivision of a face $\beta \in \Sigma$ with $\beta \prec \mu$. By assumption all these faces of Σ are TPD spaces, and therefore the subdivided faces $\widetilde{\beta}$ of $\mu_{\succeq}$ are as well. Hence we may apply Theorem I.5.4, and conclude that $\mu_{\succeq}$ is a TPD space. Thus, all of codimension 2 of Σ are TPD spaces. Proceeding inductively, we can apply Theorem I.5.4 to all the stars $\gamma_{\succeq}$ of faces $\gamma \in \Sigma$. Thus Σ is a local TPD space. ■

Corollary I.5.11. *Let (Σ, w) a d-dimensional $\mathbb{Z}$-balanced fan. Then Σ is a local TPD space over $\mathbb{Z}$ if and only if $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^{\mathbb{Z}}) = 0$ for $q \neq d$ and all $\gamma \in \Sigma$, all the weights are ± 1, and for all faces β of codimension 1, the star fans $\beta_{\succeq}$ are uniquely $\mathbb{Z}$-balanced in the sense of Definition I.3.12.*

Proof. For each face β of codimension 1 of Σ, observe that each face of dimension d of the star fan $\beta_{\succeq}$ is a linear space, hence is a TPD space over $\mathbb{Z}$. Furthermore, each star fan $\beta_{\succeq}$ has a $(d-1)$-dimensional lineality space, and we may write $\beta_{\succeq} = \Sigma_\beta \times \mathbb{R}^{d-1}$ for some "reduced star" Σ_β of dimension 1. Since $\mathbb{R}^{d-1}$ satisfies TPD over $\mathbb{Z}$, by [AP21, Proposition 3.18], the star fan $\beta_{\succeq}$ is a TPD space over $\mathbb{Z}$ if and only if Σ_β is a a TPD space over $\mathbb{Z}$. By Theorem I.4.8, this is the case if and only if Σ_β is uniquely $\mathbb{Z}$-balanced with ± 1-weights. Therefore $\beta_{\succeq}$ is a local TPD space over $\mathbb{Z}$ if and only if it is uniquely $\mathbb{Z}$-balanced with ± 1-weights. Finally, the equivalence follows from comparing with Theorem I.5.10. ■

Remark I.5.12. Passing from Theorem I.5.10 to Corollary I.5.11 is mostly dependent on the Künneth formula for the $\mathcal{F}_p^{\mathbb{Z}}$ cosheaves from [GS23]. A generalization of the Künneth formula to $\mathcal{F}_p^R$ for another ring R would also lead to a generalization of Corollary I.5.11.

Theorem I.5.10 illustrates that it would be desirable to obtain a geometric understanding of the vanishing condition for the tropical Borel–Moore homology.

Question I.5.13 (Geometry of BM homology vanishing?)**.** *Let (Σ, w) be an R-balanced d-dimensional fan. Can the fans with $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for each face $\gamma \in \Sigma$, $q \neq d$ and all p be geometrically characterized?*

We note that it is not clear whether TPD of the whole fan implies local TPD. We have not been able to construct a fan satisfying TPD such that the star of one of its faces does not.

Question I.5.14 (Global versus Local TPD)**.** *Let (Σ, w) be an R-balanced fan which satisfies TPD over R. Does $\gamma_{\succeq}$ also satisfy TPD over R for each $\gamma \in \Sigma$?*

Even assuming that $H_q^{BM}(\gamma_{\succeq}, \mathcal{F}_p^R) = 0$ for $q \neq d$, along with TPD on the whole fan (Σ, w), the proof of Theorem I.5.4 does not directly imply that Σ is a local TPD space generally. In algebraic topology, the question of going from Poincaré duality globally on a CW complex to Poincaré duality locally has been investigated using techniques from surgery and homotopy theory (see [Ran11] for an overview).

I.6 Tropical Poincaré duality for polyhedral spaces

In this section, we use the results of Section I.5 to prove that abstract tropical cycles which have charts to local TPD spaces satisfy tropical Poincaré duality. In [JRS18, Theorem 5.3], the Mayer–Vietoris argument that shows that tropical manifolds satisfy tropical Poincaré duality over $\mathbb{Z}$ is predicated on the existence of charts to fans of matroids, which are local TPD spaces over $\mathbb{Z}$. This suggests that the local TPD spaces characterized in Theorem I.5.10 are useful as building blocks for general spaces satisfying TPD. We show this in Theorem I.6.5. We refer to [JRS18] for the definitions of rational polyhedral spaces, rational polyhedral structures, as well as the tropical cohomology and Borel–Moore homology theories available on such spaces. Here we generalize these to take coefficients in an arbitrary commutative ring R, as in Definition I.3.6. Moreover, one can generalize [JRS18, Definitions 4.7-4.8] of the weight function to an arbitrary commutative ring, as in Definition I.3.11, which gives rise to a fundamental chain $\mathrm{Ch}(X, w) \in C_d^{BM}(X, \mathcal{F}_d^R)$, for $d = \dim X$.

Definition I.6.1. A rational polyhedral space X of pure dimension d with a rational polyhedral structure $\mathcal{C}$ and a weight function w is *balanced* if the fundamental chain $\mathrm{Ch}(X, w) \in C_d^{BM}(X, \mathcal{F}_d^R)$ is closed, inducing a *fundamental class* $[X, w] \in H_d^{BM}(X, \mathcal{F}_d^R)$ in tropical Borel–Moore homology. We call the triple $(X, \mathcal{C}, w)$ an *abstract tropical R-cycle.*

Abstract tropical R-cycles are the candidate spaces for *satisfying tropical Poincaré duality over R*, slightly generalizing [JRS18, Definition 4.11].

Definition I.6.2. Let X be an abstract tropical R-cycle of dimension d. The fundamental class $[X, w]$ induces a *cap product*

$$\frown [X, w] \colon H^q(X, \mathcal{F}_R^p) \to H_{d-p}^{BM}(X, \mathcal{F}_{d-p}^R)$$

between tropical cohomology and tropical Borel–Moore homology. If these are isomorphisms for all p and q, we say that X is a *tropical Poincaré duality space over R.*

Definition I.6.3. Let $(X, \mathcal{C}, w)$ be an abstract tropical R-cycle over a commutative ring R. We say that $(X, \mathcal{C}, w)$ is a *local tropical Poincaré duality space* if for each $\sigma \in \mathcal{C}$, the rational polyhedral complexes $\{\phi_\sigma(\tau) \mid \tau \in \sigma_{\succeq}\}$ are local TPD spaces over R.

Example I.6.4. Tropical manifolds, which have charts to Bergman fans of matroids, are examples of local TPD spaces over $\mathbb{Z}$ and $\mathbb{R}$, see [JRS18; JSS19].

Theorem I.6.5. *Let X be a local tropical Poincaré duality space over a principal ideal domain R. Then X satisfies tropical Poincaré duality over R.*

Proof. The two steps of the proof given in [JRS18, Proof of Theorem 5.3] carry through. Since the open stars of faces satisfy TPD over R, the first step is identical, noting that the same arguments carry through working in the category of R-modules. The induction argument given in the second step also carries through, as the same sequence of complexes can be constructed in the category of R-modules. ■

Remark I.6.6. Note that Definition I.6.3 in the case where $R = \mathbb{Z}$ is equivalent to the definition of *smooth tropical variety* given in [AP21, Definition 3.22], such that the case $R = \mathbb{Z}$ of Theorem I.6.5 is equivalent to [AP21, Theorem 3.23].

Theorem I.6.5 justifies the naming in Definition I.6.3, generalizing the relationship between local TPD spaces and TPD spaces as defined in Definition I.5.9 and Definition I.4.1. Moreover, I.5.14 about the relationship between local TPD and TPD are also applicable in this more general setting.

Question I.6.7 (Global versus Local TPD for abstract tropical cycles)**.** *Let $(X, \mathcal{C}, w)$ be an abstract tropical R-cycle satisfying TPD over R. Does $\gamma_{\succeq}$ also satisfy TPD over R for each $\gamma \in \mathcal{C}$?*

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Paper II

Cohomologically tropical varieties

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Abstract

Given the tropicalization of a complex subvariety of the torus, we define a morphism between the tropical cohomology and the rational cohomology of their respective tropical compactifications. We say that the subvariety of the torus is cohomologically tropical if this map is an isomorphism for all closed strata of the tropical compactification.

We prove that a schön subvariety of the torus is cohomologically tropical if and only if it is wunderschön and its tropicalization is a tropical homology manifold. The former property means that the open strata in the boundary of a tropical compactification are all connected and the mixed Hodge structures on their cohomology are pure of maximum possible weight; the latter property requires that, locally, the tropicalization verifies tropical Poincaré duality.

We study other properties of cohomologically tropical and wunderschön varieties, and show that in a semistable degeneration to an arrangement of cohomologically tropical varieties, the Hodge numbers of the smooth fibers are captured in the tropical cohomology of the tropicalization. This extends the results of Itenberg, Katzarkov, Mikhalkin, and Zharkov.

II

Contents

II.1 Overview

The tropicalization process transforms algebraic varieties into piecewise polyhedral objects. While losing part of the geometry, some of the invariants, such as dimension and degree, of the original variety can still be computed from its tropicalization. For the complement of a hyperplane arrangement, Zharkov shows that the tropical cohomology of the tropicalization computes the usual cohomology of the variety [Zha13]. Moreover, Hacking relates the top-weight mixed Hodge structure of a variety to the homology of its tropicalization [Hac08]. We are interested in determining for which varieties the tropicalization remembers the cohomology. Part of our motivation to study this question comes from the work of Deligne [Del97] in which he gives a Hodge-theoretic characterization of maximal degenerations of complex algebraic varieties. We are more specifically interested in determining how tropicalization is related to maximal degenerations, because of the recent connection to the SYZ conjecture by the work of Yang Li [Li20].

We introduce the relevant concepts and notation before stating our results. Let N be a lattice of rank n, M the dual lattice, and $\mathbf{T} = \mathrm{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^n$ the corresponding torus. We let $N_\mathbb{R}$ and $N_\mathbb{Q}$ denote $N \otimes_\mathbb{Z} \mathbb{R}$ and $N \otimes_\mathbb{Z} \mathbb{Q}$, respectively. Let $\mathbf{X} \subseteq \mathbf{T}$ be a non-singular subvariety of $\mathbf{T}$ and denote by $X = \mathrm{trop}(\mathbf{X})$ its tropicalization [MS15; MR18]. A unimodular fan Σ in $N_\mathbb{R}$ with support X gives rise to a complex toric variety $\mathbb{CP}_\Sigma$ and a tropical toric variety $\mathbb{TP}_\Sigma$. Taking the closures of $\mathbf{X}$ and X in $\mathbb{CP}_\Sigma$ and $\mathbb{TP}_\Sigma$, respectively, gives compactifications $\overline{\mathbf{X}}$ and $\overline{X}$. We note that the compactifications depend on the choice of the fan Σ whose support is $\mathrm{trop}(X)$, however, we have chosen not to indicate it in the notation for $\overline{\mathbf{X}}$ or $\overline{X}$. Here and elsewhere in the paper, we use bold letters for algebraic varieties and regular letters for tropical varieties.

For a complex variety $\mathbf{Z}$, we denote by $H^\bullet(\mathbf{Z})$ the cohomology ring of $\mathbf{Z}$ with coefficients in $\mathbb{Q}$. For a tropical variety Z, the k-th tropical cohomology group of Z can be defined as $H^k(Z) := \bigoplus_{p+q=k} H^{p,q}(Z)$, where $H^{p,q}(Z)$ is the (p,q)-th tropical cohomology group with $\mathbb{Q}$-coefficients introduced in [IKMZ19], see Section II.2.6. The tropical cohomology groups together form a ring $H^\bullet(Z) = \bigoplus_k H^k(Z)$, the product structure being induced by the cup product in cohomology [MZ14; GS23]. We note that the tropical cohomology of Z depends only on Z. In particular, if $Z = \mathrm{trop}(\mathbf{Z})$, no information about $\mathbf{Z}$ beyond $\mathrm{trop}(\mathbf{Z})$ goes into the recipe for computing $H^\bullet(\mathrm{trop}(\mathbf{Z}))$.

The question addressed in this paper can be informally stated as follows: Under which conditions can the cohomology of $\mathbf{X}$ be related to the tropical cohomology of $\mathrm{trop}(\mathbf{X})$?

Let $\mathbf{X}$, Σ and X be as above, with $\overline{\mathbf{X}}$ and $\overline{X}$ the corresponding compactifications. We define τ^* to be the ring homomorphism $\tau^*\colon H^\bullet(\overline{X}) \to H^\bullet(\overline{\mathbf{X}})$ between the cohomologies of $\overline{X}$ and $\overline{\mathbf{X}}$, defined by composing the isomorphism $H^{k,k}(\overline{X}) \cong A^k(\mathbb{CP}_\Sigma)$, proved in [AP21], with the cycle class map $A^k(\mathbb{CP}_\Sigma) \to H^{2k}(\mathbb{CP}_\Sigma)$ and the pullback morphism on cohomology associated to the embedding $\overline{\mathbf{X}} \hookrightarrow \mathbb{CP}_\Sigma$. The groups $H^{p,q}(\overline{X})$ are sent to zero by τ^* for $p \neq q$. We refer to Section II.3 for more details.

In the following, we will use the map τ^* not only on $\overline{X}$ and $\overline{\mathbf{X}}$, but also on some of their subvarieties: the toric varieties $\mathbb{CP}_\Sigma$ and $\mathbb{TP}_\Sigma$ are endowed with natural stratifications induced by the cone structure of Σ. Each cone $\sigma \in \Sigma$ gives rise to the torus orbits $\mathbf{T}^\sigma$ and $N_\mathbb{R}^\sigma$ in $\mathbb{CP}_\Sigma$ and $\mathbb{TP}_\Sigma$, respectively, with corresponding lattice N^σ. The closures in $\mathbb{CP}_\Sigma$ and $\mathbb{TP}_\Sigma$ of these orbits are denoted by $\mathbb{CP}_\Sigma^\sigma$ and $\mathbb{TP}_\Sigma^\sigma$, respectively, and are isomorphic to the complex and tropical toric varieties associated to the star fan Σ^σ of σ in Σ. Intersection with these strata induce a stratification of $\overline{\mathbf{X}}$ and $\overline{X}$. We denote by $\mathbf{X}^\sigma = \overline{\mathbf{X}} \cap \mathbf{T}^\sigma$ and $X^\sigma = \overline{X} \cap N_\mathbb{R}^\sigma$ the stratum associated to $\sigma \in \Sigma$, and by $\overline{\mathbf{X}}^\sigma$ and $\overline{X}^\sigma$ their closures in $\overline{\mathbf{X}}$ and $\overline{X}$, respectively. The stratum $\mathbf{X}^\sigma$ is a closed subvariety of the torus $\mathbf{T}^\sigma$ and its tropicalization coincides with X^σ. Moreover, the star fan Σ^σ is a unimodular fan with support X^σ. We thus obtain a morphism $H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ that we also denote by τ^*.

Definition II.1.1. Let $\mathbf{X} \subseteq \mathbf{T}$ be a subvariety, Σ a unimodular fan with support $X = \operatorname{trop}(\mathbf{X})$, and $\overline{\mathbf{X}}$ and $\overline{X}$ the corresponding compactifications. We say that $\mathbf{X}$ is *cohomologically tropical with respect to* Σ if the induced maps $\tau^* \colon H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ are isomorphisms for all $\sigma \in \Sigma$.

We show that the property of being cohomologically tropical for *schön* subvarieties of tori does not depend on the chosen unimodular fan. Recall from [Tev07; Hac08] that a subvariety $\mathbf{X} \subseteq \mathbf{T}$ is *schön* if for some, equivalently for any, unimodular fan Σ of support $\operatorname{trop}(\mathbf{X})$, the open strata $\mathbf{X}^\sigma$, $\sigma \in \Sigma$, of the corresponding compactification are all non-singular. It also implies that the compactification $\overline{\mathbf{X}}$ is non-singular, and that $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$ is a simple normal crossing divisor.

Theorem II.4.4. *Suppose that the subvariety* $\mathbf{X} \subseteq \mathbf{T}$ *is schön and let* $X = \operatorname{trop}(\mathbf{X})$ *be its tropicalization. The following are equivalent.*

1. *There exists a unimodular fan* Σ *with support* X *such that* $\mathbf{X}$ *is cohomologically tropical with respect to* Σ.

2. *For any unimodular fan* Σ *with support* X*,* $\mathbf{X}$ *is cohomologically tropical with respect to* Σ.

Such a schön subvariety $\mathbf{X} \subseteq \mathbf{T}$ will be called *cohomologically tropical.* For example, the linear subspaces in $\mathbb{CP}^n$, restricted to the torus, form a family of cohomologically tropical subvarieties. These very affine varieties are complements of hyperplane arrangements, see Section II.8.2. A generalization is given in [Sch21] in which Schock defines *quasilinear subvarieties* of tori as those having a tropicalization which is tropically shellable in the language of [AP21]. He shows that these subvarieties are necessarily schön. It follows from his results that quasilinear subvarieties of tori are cohomologically tropical.

We now introduce a class of subvarieties $\mathbf{X} \subseteq \mathbf{T}$ with cohomology amenable to a tropical description using the notion of mixed Hodge structures, see Section II.2.4.

Definition II.1.2. A non-singular subvariety $\mathbf{X} \subseteq \mathbf{T}$ of the torus is called *wunderschön with respect to a unimodular fan* Σ with support $\mathrm{trop}(\mathbf{X})$ if all the open strata $\mathbf{X}^\sigma$ of the corresponding compactification $\overline{\mathbf{X}}$ are non-singular and connected, and the mixed Hodge structure on $H^k(\mathbf{X}^\sigma)$ is pure of weight $2k$ for each k.

In particular, a point in the torus is wunderschön. It follows from the preceding discussion that if $\mathbf{X}$ is wunderschön, it is schön. Therefore, if $\overline{\mathbf{X}}$ is the compactification with respect to a unimodular fan Σ, the boundary $\overline{\mathbf{X}} \setminus \mathbf{X}$ is a strict normal crossing divisor.

We prove that the property of being wunderschön is independent of the fan, and that the cohomology of a wunderschön variety is divisorial in the sense of Section II.5.

Theorem II.4.5. *Suppose that the subvariety* $\mathbf{X} \subseteq \mathbf{T}$ *is wunderschön with respect to some unimodular fan. Then* $\mathbf{X}$ *is wunderschön with respect to any unimodular fan with support* $X = \mathrm{trop}(\mathbf{X})$*.*

Theorem II.5.1. *Let* $\mathbf{X} \subseteq \mathbf{T}$ *be a wunderschön subvariety. Let* $\overline{\mathbf{X}}$ *be the compactification of* $\mathbf{X}$ *with respect to a unimodular fan* Σ *with support* $X = \mathrm{trop}(\mathbf{X})$*. Then the cohomology of* $\overline{\mathbf{X}}$ *is divisorial and generated by irreducible components of* $\overline{\mathbf{X}} \setminus \mathbf{X}$*.*

A tropical variety X is called a *tropical homology manifold* if any open subset in X verifies tropical Poincaré duality. For a tropical variety which is the support of a tropical fan, this amounts to the property that for some, equivalently for any, rational unimodular fan Σ of support X, the corresponding open strata X^σ verify tropical Poincaré duality for all $\sigma \in \Sigma$. In particular this implies that, for any unimodular fan Σ of support X, any open subset of the corresponding tropical compactification $\overline{X}$ verifies tropical Poincaré duality.

A tropical fanfold X is called *Kähler* if for some, equivalently for any, quasi-projective unimodular fan Σ with support X, and for any $\sigma \in \Sigma$, the Chow ring $A^\bullet(\Sigma^\sigma)$ verifies the Kähler package, that is, Poincaré duality, hard Lefschetz theorem and Hodge-Riemann bilinear relations. Here, for a unimodular fan Σ, the Chow ring $A^\bullet(\Sigma)$ coincides with the Chow ring of the corresponding toric variety $\mathbb{CP}_\Sigma$.

We have the following main theorem on characterization of cohomologically tropical subvarieties of tori.

Theorem II.6.1. *Let* $\mathbf{X} \subseteq \mathbf{T}$ *be a schön subvariety with tropicalization* $X = \mathrm{trop}(\mathbf{X})$*. Then the following statements are equivalent.*

1. $\mathbf{X}$ *is wunderschön and* X *is a tropical homology manifold,*

2. $\mathbf{X}$ *is cohomologically tropical.*

Moreover, if any of these statements holds, then X *is Kähler.*

We deduce the following result from the above theorem.

Theorem II.6.2 (Isomorphism of cohomology on open strata)**.** *Suppose that* $\mathbf{X} \subseteq \mathbf{T}$ *is schön and cohomologically tropical. Let* Σ *be any unimodular fan with support* $X = \mathrm{trop}(\mathbf{X})$*. Then we obtain isomorphisms*

$$\tau^* \colon H^k(X^\sigma) \xrightarrow{\sim} H^k(\mathbf{X}^\sigma)$$

for all $\sigma \in \Sigma$ *and all* k*.*

Going beyond cohomologically tropical subvarieties of tori, and following the work of [IKMZ19], one can ask the following question. Which families $\mathfrak{X}_t$ of complex projective varieties over the complex disk degenerating at $t = 0$ have the property that the tropical cohomology of their tropical limit captures the Hodge numbers of a generic fiber in the family?

In Theorem II.7.1, we weaken the condition given in [IKMZ19] by showing that it suffices to ask the open components of the central fiber to be cohomologically tropical and schön. By Theorem II.6.1, this is equivalent to asking the maximal dimensional strata to be wunderschön and their tropicalizations to be tropical homology manifolds.

More precisely, let $\pi \colon \mathfrak{X} \to D^*$ be an algebraic family of non-singular algebraic subvarieties in $\mathbb{CP}^n$ parameterized by a punctured disk D^* and with fiber $\mathfrak{X}_t$ over $t \in D^*$. Let $Z \subseteq \mathbb{TP}^n$ be the tropicalization of the family.

By Mumford's proof of the semistable reduction theorem [KKMS06], we find a triangulation of Z (possibly after a base change) such that the extended family $\pi \colon \overline{\mathfrak{X}} \to D$ is regular and the fiber over zero $\mathfrak{X}_0$ is reduced and a simple normal crossing divisor. Note that since the extended family is obtained by taking the closure in a toric degeneration of $\mathbb{CP}^n$, each open stratum in $\mathfrak{X}_0$ will be naturally embedded in an algebraic torus.

We say that a tropical variety is a tropical homology manifold if all of its local tropical fanfolds verify tropical Poincaré duality.

Theorem II.7.1. *Let* $\pi \colon \mathfrak{X} \to D^*$ *be an algebraic family of subvarieties in* $\mathbb{CP}^n$ *parameterized by the punctured disk and let* $\pi \colon \overline{\mathfrak{X}} \to D$ *be a semistable extension. If the tropicalization* $Z \subseteq \mathbb{TP}^n$ *is a tropical homology manifold and all the open strata in* $\mathfrak{X}_0$ *are wunderschön, then* $H^{p,q}(Z)$ *is isomorphic to the associated graded piece* W_{2p}/W_{2p-1} *of the weight filtration in the limiting mixed Hodge structure* $H^{p+q}_{\lim}$*. The odd weight graded pieces in* $H^{p+q}_{\lim}$ *are all vanishing.*

Moreover, for $t \in D^*$*, we have* $\dim H^{p,q}(\mathfrak{X}_t) = \dim H^{p,q}(Z)$*, for all non-negative integers* p *and* q*.*

Degenerations appearing in the previous theorem are necessarily maximal in the sense of [Del97], see Section II.8.4 for more discussion of this connection.

Theorem II.7.2. *Notations as above, the family* $\mathfrak{X} \to D^*$ *is maximally degenerate.*

We refer to the earlier work of Gross-Siebert [GS10] on integral affine manifolds with singularities and degenerations to arrangements of complete toric varieties, the work of Ruddat [Rud10] on non-necessarily maximal degenerations of Calabi-Yau varieties, and [HK12; KS12; KS16; Rud21] for other interesting results connecting the topology of tropicalizations to the Hodge theory of nearby fibers.

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II.2 Preliminaries

II.2.1 Subvarieties of the torus and tropicalization

We briefly recall the tropicalization of subvarieties of tori. Let N be a lattice of rank n, M its dual, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, and $\mathbf{T} = \mathbf{T}_N = \mathrm{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^n$. Let $\mathbf{X}$ be a d-dimensional subvariety of the torus $\mathbf{T}$, so that $\mathbf{X} = \mathbf{V}(I)$ for an ideal $I \subseteq \mathbb{C}[M]$. The tropicalization of $\mathbf{X}$ can be described using initial ideals, see e.g. [MS15, Section 3.2],

$$\mathrm{trop}(\mathbf{X}) = \{w \in N_{\mathbb{R}} \mid \mathrm{in}_w(I) \neq \langle 1 \rangle\}.$$

A d-dimensional fan Σ is *weighted* if it comes equipped with a weight function $\mathbf{wt}\colon \Sigma_d \to \mathbb{Z}$ where Σ_d denotes the d-dimensional cones. A *tropical fan* is a weighted fan which is pure dimensional and which satisfies the balancing condition in tropical geometry [MS15, Section 3.3]. A *fanfold* is a subset of $N_{\mathbb{R}}$ which is the support of a rational fan, and it is a *tropical fanfold* if it is the support of a tropical fan.

The tropicalization $X \coloneqq \mathrm{trop}(\mathbf{X})$ is a tropical fanfold, and any fan structure on $\mathrm{trop}(\mathbf{X})$ is equipped with a weight function $\mathbf{wt}_{\mathbf{X}}$ induced by $\mathbf{X}$. If Σ is a rational fan in $N_{\mathbb{R}}$ of support X and η is some facet of Σ, then for a generic point w in the relative interior of η, the variety $\mathbf{V}(\mathrm{in}_w(I))$ is a union of translates of torus orbits. Then $\mathbf{wt}_{\mathbf{X}}(\eta)$ is equal to the number of such torus orbit translates counted with multiplicity. This number is invariant for generic choices of points in the relative interior of η. The tropicalization endowed with the weight function $\mathbf{wt}_{\mathbf{X}}$ satisfies the balancing condition and thus is a tropical fanfold in $N_{\mathbb{R}}$ [MS15, Section 3.4].

II.2.2 Tropical compactifications of complex varieties

We now briefly review the notion of *tropical compactifications* introduced in [Tev07]. Let Σ be a fan in $N_{\mathbb{R}}$, and $\mathbb{CP}_\Sigma$ the associated toric variety. There is a bijection between cones in Σ and torus orbits in $\mathbb{CP}_\Sigma$. For each $\sigma \in \Sigma$, we denote by $\mathbf{T}^\sigma$ the corresponding torus orbit. The closure $\overline{\mathbf{T}}^\sigma$ is the disjoint union $\bigsqcup_{\gamma \supseteq \sigma} \mathbf{T}^\gamma$, for cones $\gamma \in \Sigma$ containing σ. For $\mathbf{X} \subseteq \mathbf{T}$ a subvariety and its closure $\overline{\mathbf{X}}$ in $\mathbb{CP}_\Sigma$, we have $\overline{\mathbf{X}} = \bigsqcup_{\sigma \in \Sigma} \mathbf{T}^\sigma \cap \overline{\mathbf{X}}$. We denote the stratum $\mathbf{T}^\sigma \cap \overline{\mathbf{X}}$ by $\mathbf{X}^\sigma$, and its closure by $\overline{\mathbf{X}}^\sigma$. Note that, $\overline{\mathbf{X}}^\sigma = \bigsqcup_{\gamma \supseteq \sigma} \mathbf{X}^\gamma$.

For Σ a unimodular fan with support equal to $X = \operatorname{trop}(\mathbf{X})$, the closure $\overline{\mathbf{X}}$ of $\mathbf{X}$ in $\mathbb{CP}_\Sigma$ is compact, giving a *tropical compactification* [Tev07, Proposition 2.3]. Moreover for such a Σ, the compactification $\overline{\mathbf{X}}$ of $\mathbf{X}$ in $\mathbb{CP}_\Sigma$ is said to be *schön* if the torus action $\mathbf{T} \times \overline{\mathbf{X}} \to \mathbb{CP}_\Sigma$ is non-singular and surjective, in which case $\overline{\mathbf{X}}$ is non-singular, and the boundary $\mathbf{D} := \overline{\mathbf{X}} \smallsetminus \mathbf{X}$ is a simple normal crossing divisor [Tev07, Theorem 1.2]. The compactification $\overline{\mathbf{X}}$ is schön if and only if $\mathbf{X}^\sigma$ is non-singular for each $\sigma \in \Sigma$ [Hac08, Lemma 2.7]. If $\mathbf{X}$ admits a schön compactification, then any unimodular fan with support equal to X will provide a schön compactification [LQ11, Theorem 1.5], and in this case we will say that $\mathbf{X}$ is *schön*.

Example II.2.1. For $f = \sum_{I \in \Delta(f)} a_I x^I \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ a Laurent polynomial, it is pointed out in [Tev07, p. 1088] that the very affine hypersurface $\mathbf{X} = V(f)$ being schön is equivalent to the condition that f is *non-degenerate (with respect to its Newton Polytope)*, a concept studied in [Var76a; Var76b] and [Kou76]. For each face $\gamma \in \Delta(f)$ of the Newton Polytope of f, one defines $f_\gamma = \sum_{I \in \gamma} a_I \mathbf{x}^I$. Then f is *non-degenerate* if, for all $\gamma \in \Delta(f)$, the polynomials

$$x_1 \frac{\partial f_\gamma}{\partial x_1}, \ldots, x_n \frac{\partial f_\gamma}{\partial x_n}$$

share no common zero in $(\mathbb{C}^*)^n$. This implies that $\mathbf{X} = V(f)$ is schön by for instance [Var76b, Lemma 10.3].

II.2.3 Canonical compactifications of tropical varieties

Let Σ be a rational fan in $N_{\mathbb{R}}$. The dimension of a cone σ will be denoted by $|\sigma|$, and we denote by Σ_k the set of cones of Σ of dimension k. The unique cone of dimension 0 is denoted $\underline{0}$. Let γ, δ be two faces of Σ. We write $\gamma \preccurlyeq \delta$ if γ is a face of δ. For $\delta \in \Sigma$ a cone, the saturated sublattice parallel to δ is denoted N_δ, and the quotient lattice N/N_δ is denoted N^δ, with quotient maps $\pi^\delta \colon N \to N^\delta$ and $\pi^\delta \colon N_{\mathbb{R}} \to N^\delta_{\mathbb{R}}$. Furthermore, the *star* at δ is the fan Σ^δ in $N^\delta_{\mathbb{R}}$ whose cones are given by $\{\pi^\delta(\sigma) \mid \delta \preccurlyeq \sigma\}$.

We briefly review the construction of tropical toric varieties, referring to [MS15, Chapter 6.2] for a detailed construction. Let $\mathbb{T} = \mathbb{R} \cup \{+\infty\}$ denote the tropical semi-field. Denote by $\sigma^\vee$ the semigroup of element of $M_{\mathbb{R}}$ which are nonnegative on σ. For each $\sigma \in \Sigma$, one defines $U_\sigma^{\mathrm{trop}} := \operatorname{Hom}_{\mathrm{semigroup}}(\sigma^\vee \cap M, \mathbb{T})$, which can be identified with the set $\bigsqcup_{\delta \preccurlyeq \sigma} N^\delta_{\mathbb{R}}$. We equip U_σ^{trop} with the subset topology of the product topology on the infinite product $\mathbb{T}^{\sigma^\vee \cap M}$. For σ unimodular, U_σ^{trop} is isomorphic to $\mathbb{R}^{n-|\sigma|} \times \mathbb{T}^{|\sigma|}$. For $\delta \preccurlyeq \sigma$, the inclusion identifies U_δ^{trop} as an open subset of U_σ^{trop}. The *tropical toric variety* $\mathbb{TP}_\Sigma$ associated to Σ is the space given by gluing the U_σ^{trop} along common faces, with underlying set $\bigsqcup_{\sigma\in\Sigma} N_{\mathbb{R}}^\sigma$.

Let Σ be a fan with support X. The *canonical compactification* $\overline{X}$ of X relative to the fan Σ is the closure of X as a subset of its tropical toric variety $\mathbb{TP}_\Sigma$. Furthermore, $\overline{X}$ has a cellular structure, which we denote $\overline{\Sigma}$. See [AP21, Section 2] for details. For any cone $\sigma \in \Sigma$, we denote by X^σ the fanfold associated to Σ^σ. The canonical compactification $\overline{X}^\sigma$ of X^σ is canonically isomorphic to the closure of X^σ when considered as a subset in $N_{\mathbb{R}}^\sigma \subseteq \mathbb{TP}_\Sigma$, and we will denote this compactified fanfold by $\overline{X}^\sigma$, when Σ is understood from the context. Moreover, there is an inclusion of canonical compactifications $i\colon \overline{X}^\sigma \hookrightarrow \overline{X}^\delta$ for $\delta \preccurlyeq \sigma$.

When $X = \mathrm{trop}(\mathbf{X})$ the tropical canonical compactification $\overline{X}$ relative to any fan Σ with support X is the same as the extended tropicalization of the closure $\overline{\mathbf{X}} \subseteq \mathbb{CP}_\Sigma$ in the sense of [Kaj08] and [Pay09, Section 3].

II.2.4 Mixed Hodge structures

Keeping the notation from Section II.2.2, let $\mathbf{X} \subseteq \mathbf{T}$ be a non-singular subvariety, and Σ a unimodular fan supported on the tropicalization $X = \mathrm{Trop}(\mathbf{X})$, so that we obtain a tropical compactification $\overline{\mathbf{X}}$ of $\mathbf{X}$. Moreover suppose that the boundary $\mathbf{D} := \overline{\mathbf{X}} \smallsetminus \mathbf{X}$ is a simple normal crossing divisor. We have that $\mathbf{D} = \bigcup_{\zeta\in\Sigma_1} \overline{\mathbf{X}}^\zeta$.

By [Del71, Section 3], the *logarithmic de Rham complex* $\Omega^\bullet_{\overline{\mathbf{X}}}(\log \mathbf{D})$ induces an isomorphism
$$H^k(\mathbf{X};\mathbb{Q}) \cong \mathbb{H}^k(\overline{\mathbf{X}}; \Omega^\bullet_{\overline{\mathbf{X}}}(\log \mathbf{D})),$$
for each k. Moreover, there is a *weight filtration* $W_\bullet$ on the logarithmic de Rham complex, which gives a mixed Hodge structure on $H^k(\mathbf{X})$. This is given by the *Deligne weight spectral sequence*
$$E_1^{-p,q} = H^{q-2p}\Big(\bigsqcup_{\sigma\in\Sigma_p} \overline{\mathbf{X}}^\sigma\Big) \implies H^{q-p}(\mathbf{X}),$$
which degenerates on the E_2-page. Below, we display the rows $E_1^{\bullet,2k+1}$ and $E_1^{\bullet,2k}$, where the rightmost elements are in position $(0, 2k+1)$ and $(0, 2k)$, respectively.
$$\bigoplus_{\sigma\in\Sigma_k} H^1(\overline{\mathbf{X}}^\sigma) \to \bigoplus_{\delta\in\Sigma_{k-1}} H^3(\overline{\mathbf{X}}^\delta) \to \cdots \to \bigoplus_{\zeta\in\Sigma_1} H^{2k-1}(\overline{\mathbf{X}}^\zeta) \to H^{2k+1}(\overline{\mathbf{X}})$$
$$\bigoplus_{\sigma\in\Sigma_k} H^0(\overline{\mathbf{X}}^\sigma) \to \bigoplus_{\delta\in\Sigma_{k-1}} H^2(\overline{\mathbf{X}}^\delta) \to \cdots \to \bigoplus_{\zeta\in\Sigma_1} H^{2k-2}(\overline{\mathbf{X}}^\zeta) \longrightarrow H^{2k}(\overline{\mathbf{X}}).$$

All the differentials are sums of *Gysin homomorphisms* with appropriate signs. Recall that, given a unimodular fan Σ, and a pair of faces $\sigma, \delta \in \Sigma$ such that δ is a codimension one face of σ, the inclusion map $i\colon \overline{\mathbf{X}}^\sigma \to \overline{\mathbf{X}}^\delta$

induces a restriction map in cohomology $i^*\colon H^\bullet(\overline{\mathbf{X}}^\delta) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$, with dual map $i_*\colon H^\bullet(\overline{\mathbf{X}}^\sigma)^* \to H^\bullet(\overline{\mathbf{X}}^\delta)^*$. Applying the Poincaré duality for both $\overline{\mathbf{X}}^\sigma$ and $\overline{\mathbf{X}}^\delta$ gives a map $\mathrm{PD}^{-1}_{\overline{\mathbf{X}}^\delta} \circ i_* \circ \mathrm{PD}_{\overline{\mathbf{X}}^\sigma}\colon H^\bullet(\overline{\mathbf{X}}^\sigma) \to H^{\bullet+2}(\overline{\mathbf{X}}^\delta)$, called the *Gysin homomorphism* and denoted $\mathrm{Gys}_{\sigma\succ\delta}$.

Since the Deligne spectral sequence degenerates at the E_2 page, the cohomology of the rows $E_1^{\bullet,2k+1}$ and $E_1^{\bullet,2k}$ yields the following associated graded elements

$$\begin{array}{ccccc} \mathrm{Gr}^W_{2k+1}(H^{k+1}) & \mathrm{Gr}^W_{2k+1}(H^{k+2}) & \cdots & \mathrm{Gr}^W_{2k+1}(H^{2k}) & \mathrm{Gr}^W_{2k+1}(H^{2k+1}) \\ \mathrm{Gr}^W_{2k}(H^{k}) & \mathrm{Gr}^W_{2k}(H^{k+1}) & \cdots & \mathrm{Gr}^W_{2k}(H^{2k-1}) & \mathrm{Gr}^W_{2k}(H^{2k}), \end{array} \tag{II.1}$$

where $H^k := H^k(\mathbf{X})$ and $\mathrm{Gr}^W_l(H^k)$ denotes the weight l part of the mixed Hodge structure on H^k.

Recall that a mixed Hodge structure H is *pure of weight* n if $\mathrm{Gr}^W_i(H) = 0$ for $i \neq n$. A mixed Hodge structure H is *Hodge-Tate* if $\mathrm{Gr}^W_k(H)$ is of type (l,l) if $k = 2l$ and 0 for k odd, see, e.g., [Del97, p. 689].

II.2.5 Wunderschön varieties

We now consider wunderschön varieties $\mathbf{X} \subseteq \mathbf{T}$ as introduced in Definition II.1.2. As we noted previously, wunderschön varieties are schön. In addition, we have the following.

Proposition II.2.2. *If a non-singular subvariety $\mathbf{X} \subseteq \mathbf{T}$ is wunderschön with respect to Σ, then the weight function of the tropicalization $\mathbf{wt}_{\mathbf{X}}$ is equal to one on all top dimensional faces η of Σ.*

Proof. The weight $\mathbf{wt}_{\mathbf{X}}(\eta)$ is equal to the intersection multiplicity of $\overline{\mathbf{X}}$ with the toric stratum $\mathbb{CP}_{\Sigma^\eta}$. In other words, it is the number of points in the variety $\overline{\mathbf{X}}^\eta$ counted with multiplicities. Since $\mathbf{X}$ is wunderschön, the variety $\overline{\mathbf{X}}^\eta$ must consist of a single point. Hence, for all facets η we have $\mathbf{wt}_{\mathbf{X}}(\eta) = 1$. ■

A consequence of the wunderschön property is that, for each $\sigma \in \Sigma$, the even rows of the $E_2 = E_\infty$-page for $\mathbf{X}^\sigma$, taking a priori the form shown in (II.1), are in fact zero except in the leftmost position, which implies that $H^k(\mathbf{X}^\sigma) = \mathrm{Gr}^W_{2k}(H^k(\mathbf{X}^\sigma))$. Moreover, the odd rows of the E_1-page are all identically zero by the following lemma.

Lemma II.2.3. *Let $\mathbf{X} \subseteq \mathbf{T}$ be a wunderschön variety with respect to Σ. Then $H^{2k-1}(\overline{\mathbf{X}}^\sigma) = 0$ for $k = 1, \ldots, \dim(\overline{\mathbf{X}}^\sigma)$ and all $\sigma \in \Sigma$.*

Proof. The property is true for a wunderschön point. By induction on dimension, we have $H^{2k-1}(\overline{\mathbf{X}}^\sigma) = 0$ for $k = 1, \ldots, \dim(\overline{\mathbf{X}}^\sigma)$ and all cones σ except the central vertex $\underline{0}$, so that it remains to prove that $H^{2k-1}(\overline{\mathbf{X}}) = 0$ for $k = 1, \ldots, \dim(\overline{\mathbf{X}})$. For each such k, the $(2k-1)$-th row of the E_2-page of the Deligne spectral sequence is given by $E_2^{0,2k-1} = H^{2k-1}(\overline{\mathbf{X}}) = \mathrm{Gr}^W_{2k-1}(H^{2k-1}(\mathbf{X}))$ and all other terms are 0. Since $\mathbf{X}$ is wunderschön, $\mathrm{Gr}^W_{2k-1}(H^{2k-1}(\mathbf{X})) = 0$, and so $H^{2k-1}(\overline{\mathbf{X}}) = 0$. ■

Since the E_2-page is the cohomology of the E_1-page, this proves the following lemma.

Lemma II.2.4. *For* $\mathbf{X} \subseteq \mathbf{T}$ *a wunderschön variety with respect to* Σ*, and for each cone* $\sigma \in \Sigma$ *and each* k*, we have the following exact sequences*

$$0 \to H^k(\mathbf{X}^\sigma) \xrightarrow{\text{res}} \bigoplus_{\substack{\mu \succ \sigma \\ |\mu|=|\sigma|+k}} H^0(\overline{\mathbf{X}}^\mu) \xrightarrow{\text{Gys}} \bigoplus_{\substack{\nu \succ \sigma \\ |\nu|=|\sigma|+k-1}} H^2(\overline{\mathbf{X}}^\nu) \xrightarrow{\text{Gys}} \cdots$$
$$\cdots \xrightarrow{\text{Gys}} \bigoplus_{\substack{\xi \succ \sigma \\ |\xi|=|\sigma|+1}} H^{2k-2}(\overline{\mathbf{X}}^\xi) \xrightarrow{\text{Gys}} H^{2k}(\overline{\mathbf{X}}^\sigma) \to 0,$$

where res *denotes the logarithmic residue map and* Gys *denotes a signed sum of suitable Gysin maps.*

Example II.2.5 (Wunderschön curves are rational)**.** We classify wunderschön curves $\mathbf{X} \subseteq \mathbf{T}$. A tropical compactification $\overline{\mathbf{X}}$ consists of adding points to $\mathbf{X}$. Points have pure mixed Hodge structure on their cohomology. Thus, for $\mathbf{X}$ to be wunderschön with respect to a fan Σ, it is necessary that each stratum X^ζ for $\zeta \in \Sigma_1$ be connected, i.e., consists of a single point. The Deligne weight spectral sequence degenerates on the E_2 page, and is shown in Figure II.1 and Figure II.2. Note that $H^2(\mathbf{X})$ is trivial. Moreover, if $\mathbf{X}$ is wunderschön, then $H^1(\overline{\mathbf{X}}) = \mathrm{Gr}_1^W(H^1(\mathbf{X}))$ must be trivial.

-1	0	
$H^0(\bigsqcup_{\zeta\in\Sigma_1} \overline{\mathbf{X}}^\zeta) \longrightarrow$	$H^2(\overline{\mathbf{X}})$	2
0	$H^1(\overline{\mathbf{X}})$	1
0	$H^0(\overline{\mathbf{X}})$	0

Figure II.1: E_1-page from Example II.2.5

-1	0	
$\mathrm{Gr}_2^W(H^1(\mathbf{X}))$	0	2
0	$\mathrm{Gr}_1^W(H^1(\mathbf{X}))$	1
0	$\mathrm{Gr}_0^W(H^0(\mathbf{X}))$	0

Figure II.2: E_2-page from Example II.2.5

Therefore, a non-singular curve $\mathbf{X} \subseteq \mathbf{T}$ is wunderschön if and only if the curve $\overline{\mathbf{X}}$ is isomorphic to $\mathbb{CP}^1$ and it meets each toric boundary divisor of $\mathbb{CP}_\Sigma$ in only one point. We conclude that the only wunderschön (open) curves are complements of a finite set of points in a non-singular rational curve.

II.2.6 Tropical homology and cohomology

We now briefly sketch the theory of tropical homology and cohomology, and refer to [JRS18; IKMZ19; JSS19; AP20; AP21; Aks23; GS23] for details. We work with $\mathbb{Q}$-coefficients.

Let Σ be a rational fan in $N_{\mathbb{R}}$ with support X. Let $\overline{X}$ be the closure of X inside the tropical toric variety $\mathbb{TP}_{\Sigma}$. The closure $\overline{X}$ has a cellular structure $\overline{\Sigma}$ where the cells of $\overline{\Sigma}$ consist of the closures of the cones in Σ^{σ} for all $\sigma \in \Sigma$. In particular, each face of $\overline{\Sigma}$ is indexed by a pair of cones $\sigma, \gamma \in \Sigma$ satisfying $\gamma \succcurlyeq \sigma$, and denoted $C^{\sigma}_{\gamma} \in \overline{\Sigma}$. For each face $C^{\sigma}_{\gamma} \in \overline{\Sigma}$, the *p-th multi-tangent space* $\mathcal{F}_p(C^{\sigma}_{\gamma})$ (with $\mathbb{Q}$-coefficients) is defined as

$$\mathcal{F}_p(C^{\sigma}_{\gamma}) \coloneqq \sum_{\eta \succcurlyeq \gamma} \bigwedge^{p} (N_{\eta}/N_{\sigma}) \otimes \mathbb{Q} \subseteq \bigwedge^{p} N^{\sigma}_{\mathbb{Q}}.$$

Moreover, for $\alpha \preccurlyeq \beta$ two faces of $\overline{\Sigma}$, there is a map $\iota_{\beta \succcurlyeq \alpha} \colon \mathcal{F}_p(\beta) \to \mathcal{F}_p(\alpha)$, which is an inclusion if both faces lie in the same subfan Σ^{σ} for some σ, or if $\alpha = C^{\sigma}_{\eta}$ and $\beta = C^{\sigma'}_{\eta}$ with $\eta \succcurlyeq \sigma \succ \sigma'$, then $\iota_{\beta \succcurlyeq \alpha}$ is induced by the projection $N^{\sigma'} \to N^{\sigma}$. Generally, the map $\iota_{\beta \succcurlyeq \alpha}$ is defined as compositions of such inclusions and projections. Furthermore, by dualizing, we obtain the *p-th multi-cotangent spaces* $\mathcal{F}^p(\alpha)$ and reversed morphisms.

By selecting orientations for each of the cones $\alpha \in \overline{\Sigma}$, we obtain relative compatibility signs $\operatorname{sign}(\alpha, \beta) \in \{\pm 1\}$ for $\alpha \prec \beta$ with $|\beta| = |\alpha| + 1$. We may thus use the multi-tangent spaces to define a chain complex

$$C_{p,q}(\overline{\Sigma}) \coloneqq \bigoplus_{\alpha \in \overline{\Sigma}_q} \mathcal{F}_p(\alpha),$$

that is, summing over faces α of dimension q in $\overline{\Sigma}$, with differentials $\partial_q \coloneqq C_{p,q}(\overline{\Sigma}) \to C_{p,q-1}(\overline{\Sigma})$ defined component-wise as the maps $\operatorname{sign}(\alpha, \beta)\iota_{\beta \succ \alpha}$ when $\alpha \prec \beta$ and $|\beta| = |\alpha| + 1$, and defined to be 0, otherwise. Similarly, by dualizing everything, we obtain a cochain complex $C^{p,q}(\overline{\Sigma})$ for the multi-cotangent spaces.

The homology groups $H_{p,q}(\overline{\Sigma}) \coloneqq H_q(C_{p,\bullet}(\overline{\Sigma}))$ of the complex $C_{p,\bullet}(\overline{\Sigma})$ are invariants of the canonically compactified support $\overline{X}$ of the support X of the fan Σ. Therefore, we define the *tropical homology of* $\overline{X}$ as the homology $H_{p,q}(\overline{X}) \coloneqq H_q(C_{p,\bullet}(\overline{\Sigma}))$ of the complex $C_{p,\bullet}(\overline{\Sigma})$. The *tropical cohomology of* $\overline{X}$ is $H^{p,q}(\overline{X}) \coloneqq H^q(C^{p,\bullet}(\overline{\Sigma}))$.

In fact, tropical homology and cohomology can be defined for any rational polyhedral space. Moreover, there are various equivalent descriptions of tropical (co)homology in terms of cellular, singular, and sheaf theoretic terms [MZ14; IKMZ19; GS23]. For any rational polyhedral space Z, we set

$$H^k(Z) \coloneqq \bigoplus_{p+q=k} H^{p,q}(Z).$$

For example, for a fanfold X, the tropical homology is $H_{p,q}(X) = \mathcal{F}_p(\underline{0})$ if $q = 0$ and 0 otherwise, and the tropical cohomology of X is $H^{p,q}(X) = \mathcal{F}^p(\underline{0})$ if $q = 0$ and 0 otherwise [JSS19, Proposition 3.11].

If X is a tropical fanfold, the balancing condition implies the existence of a *fundamental class* $[\overline{X}] \in H_{d,d}(\overline{X})$, which induces a *cap product* $\frown$

$[\overline{X}]\colon H^{p,q}(\overline{X}) \to H_{d-p,d-q}(\overline{X})$ for each $p, q \in \{0, \dots, d\}$. When these maps are isomorphisms for all p and q, the variety $\overline{X}$ is said to satisfy *tropical Poincaré duality.*

Definition II.2.6. A tropical fanfold X is called a *tropical homology manifold* if one of the three following equivalent conditions hold:

- **There exists** a unimodular fan Σ with support equal to X such that each of the canonical compactifications $\overline{X}^\sigma$ satisfies tropical Poincaré duality, for all cones $\sigma \in \Sigma$.
- **For any** unimodular fan Σ with support equal to X, each of the canonical compactifications $\overline{X}^\sigma$ satisfies tropical Poincaré duality, for all cones $\sigma \in \Sigma$.
- Any open subset U of X satisfies tropical Poincaré duality,, i.e.,, the tropical Poincaré duality induces an isomorphism between the tropical cohomology and the tropical Borel-Moore homology of U (see [JRS18; JSS19] for details).

This definition corresponds to the notion of *tropical smoothness* in [AP21] and to *local tropical Poincaré duality spaces* in [Aks23]. The equivalence of the three statements is non-trivial and follows from Theorems 3.20, 3.23 and 7.9 of the article [AP21].

II.2.7 Chow rings of fans

We now recall some facts about the Chow ring of a fan, see for instance [AP20; AP21] for more details.

Let Σ be a unimodular fan in a vector space $N_\mathbb{R}$. The Chow ring $A^\bullet(\Sigma)$ is the quotient ring

$$A^\bullet(\Sigma) \coloneqq \mathbb{Q}[\mathrm{x}_\zeta \mid \zeta \in \Sigma_1]\big/(I + J)$$

with a variable x_ζ for each ray $\zeta \in \Sigma_1$. Here I is the ideal generated by all monomials $\mathrm{x}_{\zeta_1} \cdots \mathrm{x}_{\zeta_l}$ such that the rays $\zeta_1, \dots, \zeta_l$ do not form a cone of Σ; and J is the ideal generated by the expressions $\sum_{\zeta \in \Sigma_1} \langle m, \mathfrak{e}_\zeta \rangle \mathrm{x}_\zeta$, where $\mathfrak{e}_\zeta \in N$ is the primitive vector of the ray ζ and m ranges over elements of the dual lattice M.

For $\sigma \in \Sigma$, we define $\mathrm{x}_\sigma \coloneqq \mathrm{x}_{\zeta_1} \cdots \mathrm{x}_{\zeta_k}$, where $\zeta_1, \dots, \zeta_k$ are the rays of σ. As a vector space, $A^\bullet(\Sigma)$ is generated by x_σ, $\sigma \in \Sigma$. For a pair of cones $\delta \preccurlyeq \sigma$, there is a Gysin map $\mathrm{Gys}_{\sigma \succcurlyeq \delta}\colon A^\bullet(\Sigma^\sigma) \to A^{\bullet + |\sigma| - |\delta|}(\Sigma^\delta)$. This map is defined by mapping $\mathrm{x}_{\eta'} \in \Sigma^\sigma$ to $\mathrm{x}_\eta \mathrm{x}_{\zeta_1} \cdots \mathrm{x}_{\zeta_r}$, where η' is a face of Σ^σ, η is the corresponding face in Σ^δ, and $\zeta_1, \dots, \zeta_r$ are the rays of σ not in δ.

Since Σ is unimodular, there is an isomorphism of rings $\Phi_\Sigma\colon A^\bullet(\Sigma) \xrightarrow{\sim} A^\bullet(\mathbb{CP}_\Sigma)$ from the Chow ring of Σ to the Chow ring of the toric variety $\mathbb{CP}_\Sigma$, see e.g., [Bri96, Section 3.1]. Furthermore, the cycle class map $\mathrm{cyc}_\Sigma\colon A^\bullet(\mathbb{CP}_\Sigma) \to H^{2\bullet}(\mathbb{CP}_\Sigma)$ gives a graded ring homomorphism to cohomology, see [Ful84, Corollary 19.2]. Consider a subvariety $\mathbf{X}$ of the torus, and assume that the support of Σ is $\mathrm{Trop}(\mathbf{X})$. Let $\overline{\mathbf{X}}$ be the corresponding compactification. There is the restriction map of rings $r^*\colon H^\bullet(\mathbb{CP}_\Sigma) \to H^\bullet(\overline{\mathbf{X}})$. Composing all these homomorphisms gives a morphism of rings $\Phi\colon A^\bullet(\Sigma) \to H^{2\bullet}(\overline{\mathbf{X}})$ which maps x_σ to the class of $\overline{\mathbf{X}}^\sigma$.

In the tropical world, there is a similar map. Let X be the support of Σ and let $\overline{X}$ be the corresponding compactification. One can consider the composition

$$A^\bullet(\Sigma) \to H^{2\bullet}(\mathbb{TP}_\Sigma) \to H^{2\bullet}(\overline{X})$$

mapping x_σ to the class of $\overline{X}^\sigma$. By the Hodge isomorphism theorem [AP21, Theorem 7.1], this composition induces an isomorphism of rings $\bigoplus_k A^k(\Sigma) \xrightarrow{\sim} \bigoplus_k H^{k,k}(\overline{X})$. We define the inverse map $\Psi\colon H^\bullet(\overline{X}) \to A^{\bullet/2}(\Sigma)$ by mapping (p,q)-classes to zero if $p \neq q$. Here, by convention, $A^{k/2}(\Sigma)$ is trivial for odd k. If Σ is a tropical homology manifold, Ψ is an isomorphism by [AP21, Theorem 7.2], that is, $H^{p,q}(\overline{X})$ is trivial for $p \neq q$.

Kähler package

We recall the Kähler package for Chow rings of fans, see [AP23]. Assume Σ is tropical and quasi-projective,, i.e.,, there exists a conewise linear function f on Σ which is strictly convex in the following sense. For any $\sigma \in \Sigma$, there exists a linear map $m \in M$ such that $f - m$ is zero on σ and strictly positive on $U \setminus \sigma$ for some open neighborhood U of the relative interior of σ. To such an f, one can associate the element $L \coloneqq \sum_{\zeta\in\Sigma_1} f(\mathfrak{e}_\zeta)\mathrm{x}_\zeta \in A^1(\Sigma)$. These elements coming from strictly convex functions are called *ample classes*. Since Σ is tropical, the degree map $\deg\colon A^d(\Sigma) \to \mathbb{Q}$ mapping x_η to $\mathbf{wt}(\eta)$ for any facet η of Σ is a well-defined morphism.

The Chow ring $A^\bullet(\Sigma)$ is said to verify the Kähler package if the following holds:

- (Poincaré duality) the pairing

$$\begin{array}{ccc} A^k(\Sigma) \times A^{d-k}(\Sigma) & \to & \mathbb{Q}, \\ (a,b) & \mapsto & \deg(ab), \end{array}$$

  is perfect for any k;

- (Hard Lefschetz theorem) for any ample class L, the multiplication by L^{d-2k} induces an isomorphism between $A^k(\Sigma)$ and $A^{d-k}(\Sigma)$ for all $k \leq d/2$;

- (Hodge-Riemann bilinear relations) for any $k \leq d/2$ and any ample class L, the bilinear map

$$\begin{array}{ccc} A^k(\Sigma) \times A^k(\Sigma) & \to & \mathbb{Q}, \\ (a,b) & \mapsto & (-1)^k \deg(L^{d-2k}ab), \end{array}$$

  is positive definite on $\ker(\,\cdot L^{d-2k+1}\colon A^k(\Sigma) \to A^{d-k+1}(\Sigma))$.

A tropical fanfold X is called *Kähler* if it is a tropical homology manifold and there exists a quasi-projective unimodular fan of support X such that $A^\bullet(\Sigma^\sigma)$ verifies the Kähler package for any $\sigma \in \Sigma$. In such a case, any quasi-projective unimodular fan Σ on X verifies the previous property (cf. [AP23]).

II.2.8 Tropical Deligne resolution

Let Σ be a unimodular fan on some tropical homology manifold X. Let $\delta \preccurlyeq \sigma$ be two faces of Σ. The inclusion $i^{\mathrm{trop}}\colon \overline{X}^{\sigma} \to \overline{X}^{\delta}$ of canonically compactified fanfolds, both satisfying tropical Poincaré duality, gives a homomorphism $i_*^{\mathrm{trop}}\colon H_k(\overline{X}^{\sigma}) \to H_k(\overline{X}^{\delta})$. Applying the tropical Poincaré duality for both $\overline{X}^{\sigma}$ and $\overline{X}^{\delta}$, this gives a map $\mathrm{PD}_{\overline{X}^{\delta}}^{-1} \circ i_*^{\mathrm{trop}} \circ \mathrm{PD}_{\overline{X}^{\sigma}}\colon H^k(\overline{X}^{\sigma}) \to H^{k+2(|\sigma|-|\delta|)}(\overline{X}^{\delta})$, called the *tropical Gysin homomorphism* and denoted $\mathrm{Gys}_{\sigma \succcurlyeq \delta}^{\mathrm{trop}}$.

In [AP21, Theorem 8.1], it is shown that for a fanfold X which is a tropical homology manifold and a unimodular fan Σ with support X, there are *tropical Deligne resolutions*, i.e., exact sequences for any k,

$$0 \longrightarrow H^k(X) \longrightarrow \bigoplus_{\sigma \in \Sigma_k} H^0(\overline{X}^{\sigma}) \longrightarrow \bigoplus_{\delta \in \Sigma_{k-1}} H^2(\overline{X}^{\delta}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{\zeta \in \Sigma_1} H^{2k-2}(\overline{X}^{\zeta}) \longrightarrow H^{2k}(\overline{X}) \longrightarrow 0,$$

where the first non-zero morphism is given by integration (that is, by the evaluation of the element $\alpha \in H^k(X)$ at the canonical multivector of each face $\sigma \in \Sigma_k$), and all subsequent maps are given by the tropical Gysin homomorphisms (with appropriate signs [AP21, Section 8]).

II.3 The induced morphism on cohomology by tropicalization

The aim of this section is to define a map relating tropical cohomology to classical cohomology, as well as to prove Proposition II.3.2, which relates Gysin maps in tropical and classical cohomology.

Definition II.3.1. Let $\mathbf{X} \subseteq \mathbf{T}$ be a subvariety and Σ a unimodular fan with support $X = \mathrm{trop}(\mathbf{X})$, and $\overline{\mathbf{X}}$ and $\overline{X}$ be the compactifications of $\mathbf{X}$ and X with respect to Σ. We define

$$\tau^*\colon H^{\bullet}(\overline{X}) \to H^{\bullet}(\overline{\mathbf{X}})$$

to be the ring homomorphism defined as the composition of the maps $\Psi\colon H^{\bullet}(\overline{X}) \to A^{\bullet/2}(\Sigma)$ with $\Phi\colon A^{\bullet/2}(\overline{X}) \to H^{\bullet}(\overline{\mathbf{X}})$ from Section II.2.7.

The map τ^* is the morphism comparing the tropical and classical cohomology in order to define *cohomologically tropical* varieties in Definition II.1.1.

We will now relate the classical and tropical Gysin maps through the map τ^*. This will be useful later for comparing Deligne sequences.

Proposition II.3.2. *Let* $X = \mathrm{Trop}(\mathbf{X})$ *the be tropicalization of a subvariety* $\mathbf{X} \subseteq \mathbf{T}$*,* Σ *a unimodular fan with support* X*, with* $\sigma, \delta \in \Sigma$ *such that* δ *is a face of* σ *of codimension one, giving inclusion maps* $\overline{\mathbf{X}}^{\sigma} \to \overline{\mathbf{X}}^{\delta}$ *and* $\overline{X}^{\sigma} \to \overline{X}^{\delta}$*. Then* *the following diagram commutes:*

$$\begin{array}{ccc} H^k(\overline{X}^\sigma) & \xrightarrow[\tau^*]{} & H^k(\overline{\mathbf{X}}^\sigma) \\ \big\downarrow \mathrm{Gys}^{\mathrm{trop}}_{\sigma\succ\delta} & & \big\downarrow \mathrm{Gys}_{\sigma\succ\delta} \\ H^{k+2}(\overline{X}^\delta) & \xrightarrow{\tau^*} & H^{k+2}(\overline{\mathbf{X}}^\delta). \end{array}$$

Proof. Expanding the definition of τ^*, we obtain the following diagram

$$\begin{array}{ccccc} H^\bullet(\overline{X}^\sigma) & \xrightarrow{\Psi} & A^{\bullet/2}(\Sigma^\sigma) & \xrightarrow{\Phi} & H^\bullet(\overline{\mathbf{X}}^\sigma) \\ \mathrm{Gys}^{\mathrm{trop}}_{\sigma\succ\delta}\big\downarrow & & \big\downarrow \mathrm{Gys}_{\sigma\succ\delta} & & \big\downarrow \mathrm{Gys}_{\sigma\succ\delta} \\ H^{\bullet+2}(\overline{X}^\delta) & \xrightarrow{\Psi} & A^{\bullet/2+1}(\Sigma^\delta) & \xrightarrow{\Phi} & H^{\bullet+2}(\overline{\mathbf{X}}^\delta). \end{array}$$

The first square is commutative by [AP20, Remark 3.15], in light of [AP21, Theorem 7.1]. The commutativity of the second square follows from the functoriality of the cycle class map in light of [Bri96, Section 3.2] and [Ful84, Section 19.2]. ■

Remark II.3.3. Let $\mathbf{X} \subseteq \mathbf{T}_N$ and $\mathbf{X}' \subseteq \mathbf{T}_{N'}$ be two non-singular subvarieties of tori associated to two lattices N and N', with X and X' the corresponding tropicalizations, and two unimodular fans Σ and Σ' with supports X and X', respectively.

Assume there exists a morphism of lattices $\phi\colon N \to N'$ which takes cones of Σ to cones of Σ' such that the induced map $\phi|_X\colon X \to X'$ is surjective. This makes the induced morphism of toric varieties $f\colon \mathbb{CP}_\Sigma \to \mathbb{CP}_{\Sigma'}$ proper [Ful93, Section 2.4]. We denote by $f^{\mathrm{trop}}\colon \mathbb{TP}_\Sigma \to \mathbb{TP}_{\Sigma'}$ the induced morphism on tropical toric varieties.

Furthermore, suppose that $f(\mathbf{X}) = \mathbf{X}'$. Since $\overline{\mathbf{X}}$ is compact we have that $f(\overline{\mathbf{X}}) = \overline{f(\mathbf{X})} = \overline{\mathbf{X}}'$. This also gives $f^{\mathrm{trop}}(\overline{X}) = \overline{X}'$ for the canonical compactifications of X and X' with respect to Σ and Σ'. One can then prove the commutativity of the following diagram

$$\begin{array}{ccc} H^\bullet(\overline{X}') & \xrightarrow{\tau^*} & H^\bullet(\overline{\mathbf{X}}') \\ f^{\mathrm{trop},*}\big\downarrow & & \big\downarrow f^* \\ H^\bullet(\overline{X}) & \xrightarrow[\tau^*]{} & H^\bullet(\overline{\mathbf{X}}). \end{array}$$

Proposition II.3.4. *Let $\mathbf{X} \subseteq \mathbf{T}$ be a subvariety of complex dimension d and Σ a unimodular fan with support $X = \mathrm{trop}(\mathbf{X})$, and $\overline{\mathbf{X}}$ and $\overline{X}$ be the compactifications of $\mathbf{X}$ and X with respect to Σ. Suppose $\overline{X}$ satisfies tropical Poincaré duality and $\overline{\mathbf{X}}$ is non-singular. Then $\tau^*\colon H^\bullet(\overline{X}) \to H^\bullet(\overline{\mathbf{X}})$ is injective.*

Proof. Both maps $\Psi\colon H^{2d}(\overline{X}) \to A^d(\Sigma)$ and $\Phi\colon A^d(\Sigma) \to H^{2d}(\overline{\mathbf{X}})$ commute with the corresponding degree maps. Now for both tropical and classical cohomology, the fact that the products induce perfect pairings implies that τ^* is injective. ■

II.4 Irrelevance of fan

To be schön, wunderschön, cohomologically tropical, Kähler, or a tropical homology manifold are all properties of the form "there exists a fan Σ such that a specific property holds" with some restriction on the fan, as unimodularity for instance. Informally, we say that such a property is *fan irrelevant* if we can replace "there exists a unimodular fan" by "for any unimodular fan" (this is strongly linked with the notion of shellable properties in [AP21]). It is already known that to be schön, Kähler or a tropical homology manifold is fan irrelevant. In this section we prove Theorems II.4.4 and II.4.5 about the fan irrelevance of being cohomologically tropical and wunderschön. We begin with a lemma.

Lemma II.4.1. *Suppose a schön subvariety $\mathbf{X} \subseteq \mathbf{T}$ is cohomologically tropical. Then the tropicalization $X = \mathrm{trop}(\mathbf{X})$ is a tropical homology manifold.*

Proof. Let Σ be a unimodular fan whose support is $\mathrm{trop}(\mathbf{X})$. It follows that the cohomology groups $H^\bullet(\overline{X}^\sigma)$ are all isomorphic to the cohomology groups $H^\bullet(\overline{\mathbf{X}}^\sigma)$, and so they verify Poincaré duality. We infer that X is a tropical homology manifold. ■

Let $\mathbf{X}$ be a schön subvariety of the torus which is cohomologically tropical. It follows from the previous lemma and the fan irrelevance of being a tropical homology manifold that all the cohomology groups $H^{p,q}(\overline{X})$ are vanishing provided that $p \neq q$, for the canonical compactification $\overline{X}$ of X with respect to any unimodular fan with support X.

Let Σ be a unimodular fan with support the fanfold X, and let σ be a cone in Σ of dimension at least two. Let Σ' be the barycentric star subdivision of Σ obtained by star subdividing σ, see e.g. [Wło03; AP21]. Denote by ρ the new ray in Σ'. Let $\overline{X}$ and $\overline{X}'$ be the compactifications of X with respect to Σ and Σ', respectively.

The following theorem provides a description of the Chow ring of Σ' in terms of the Chow rings of Σ and Σ^σ.

Theorem II.4.2 (Keel's lemma)**.** *Let $\mathfrak{J}$ be the kernel of the surjective map $\mathrm{i}^*_{\underline{0}\prec\sigma} \colon A^\bullet(\Sigma) \to A^\bullet(\Sigma^\sigma)$ and let*

$$P(T) \coloneqq \prod_{\substack{\zeta\prec\sigma\\|\zeta|=1}} (\mathrm{x}_\zeta + T).$$

There is an isomorphism of Chow groups given by the map

$$\chi\colon A^\bullet(\Sigma)[T]/(\mathfrak{J}T + P(T)) \xrightarrow{\sim} A^\bullet(\Sigma')$$

which sends T to $-\mathrm{x}_\rho$ and which verifies

$$\forall \zeta \in \Sigma_1, \qquad \chi(\mathrm{x}_\zeta) = \begin{cases} \mathrm{x}_\zeta + \mathrm{x}_\rho & \text{if } \zeta \prec \sigma, \\ \mathrm{x}_\zeta & \text{otherwise.} \end{cases}$$

In particular this gives a vector space decomposition of $A^\bullet(\Sigma')$ *as*

$$A^\bullet(\Sigma') \cong A^\bullet(\Sigma) \oplus A^{\bullet-1}(\Sigma^\sigma)T \oplus \cdots \oplus A^{\bullet-|\sigma|+1}(\Sigma^\sigma)T^{|\sigma|-1}. \qquad \text{(II.2)}$$

In addition, if X *is the tropicalization of a schön subvariety* $\mathbf{X} \subseteq \mathbf{T}$*, and* $\overline{\mathbf{X}}$ *and* $\overline{\mathbf{X}}'$ *are compactifications of* $\mathbf{X}$ *with respect to* Σ *and* Σ'*, respectively, then we have an isomorphism*

$$H^\bullet(\overline{\mathbf{X}}') \cong H^\bullet(\overline{\mathbf{X}})[T]/(\mathfrak{J}T + P(T)),$$

and the decomposition

$$H^\bullet(\overline{\mathbf{X}}') \cong H^\bullet(\overline{\mathbf{X}}) \oplus H^{\bullet-1}(\overline{\mathbf{X}}^\sigma)T \oplus \cdots \oplus H^{\bullet-|\sigma|+1}(\overline{\mathbf{X}}^\sigma)T^{|\sigma|-1}. \qquad \text{(II.3)}$$

Here, by an abuse of notation, the variable T *denotes the image of* $-\mathrm{x}_\rho$ *in* $H^2(\overline{\mathbf{X}}')$ *for the induced map* $A^\bullet(\Sigma') \to H^\bullet(\overline{\mathbf{X}}')$*,* $\mathfrak{J}$ *is the kernel of* $H^\bullet(\overline{\mathbf{X}}) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$*, and* $P(T)$ *is the image of* $\prod_{\substack{\zeta \prec \sigma \\ |\zeta|=1}} (\mathrm{x}_\zeta + T)$ *in* $H^\bullet(\overline{\mathbf{X}})[T]$ *under the map* $A^\bullet(\Sigma) \to H^\bullet(\overline{\mathbf{X}})$*.*

Decomposition (II.2), for instance, means that for any $1 \leq k \leq |\sigma|$, we have a natural injective map

$$A^\bullet(\Sigma^\sigma) \hookrightarrow A^\bullet(\Sigma'^\rho) \xhookrightarrow{-\,\mathrm{Gys}_{\rho \succ \underline{0}}} A^{\bullet+1}(\Sigma') \xhookrightarrow{T^{k-1}} A^{\bullet+k}(\Sigma').$$

The piece $A^\bullet(\Sigma^\sigma)T^k$ in the above decomposition then denotes the image of the above map. We refer to [Kee92] and [AP23] for more details and the proof.

Two unimodular fans with the same support are called *elementary equivalent* if one can be obtained from the other by a barycentric star subdivision. The *weak equivalence* between unimodular fans with the same support is then defined as the transitive closure of the elementary equivalence relation. We will need the weak factorization theorem, stated as follows.

Theorem II.4.3 (Weak factorization theorem [Mor96; Wło97])**.** *Two unimodular fans with the same support are always weakly equivalent.*

We are now in a position to prove the independence of being cohomologically tropical from the chosen fan for schön varieties.

Theorem II.4.4. *Suppose that the subvariety* $\mathbf{X} \subseteq \mathbf{T}$ *is schön and let* $X = \mathrm{trop}(\mathbf{X})$ *be its tropicalization. The following are equivalent.*

II.4.4.1. *There exists a unimodular fan* Σ *with support* X *such that* $\mathbf{X}$ *is cohomologically tropical with respect to* Σ*.*

II.4.4.2. *For any unimodular fan* Σ *with support* X*,* $\mathbf{X}$ *is cohomologically tropical with respect to* Σ*.*

Proof. Suppose that the subvariety $\mathbf{X}$ of the torus $\mathbf{T}$ is schön. Let $X = \mathrm{trop}(\mathbf{X})$. Let Σ be a unimodular fan with support X such that $\mathbf{X}$ is cohomologically tropical with respect to Σ. Let Σ' be a second unimodular fan with support X. We need to prove that $\mathbf{X}$ is cohomologically tropical with respect to Σ'. By the weak factorization theorem, it will be enough to assume that Σ and Σ' are elementary equivalent.

We consider the compactifications $\overline{\mathbf{X}}'$ and $\overline{X}'$ of $\mathbf{X}$ and X with respect to Σ', and those with respect to Σ by $\overline{\mathbf{X}}$ and $\overline{X}$.

Consider first the case where Σ' is obtained as a barycentric star subdivision of Σ. Denote by σ the cone of Σ which has been subdivided and by ρ the new ray of Σ'.

We start by explaining the proof of the isomorphism $H^\bullet(\overline{X}') \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}')$. We use the notation preceding Theorem II.4.3. By Keel's lemma, we get

$$A^\bullet(\Sigma') \cong A^\bullet(\Sigma)[T]/(\mathfrak{J}T+P(T)) \qquad \text{and} \qquad H^\bullet(\overline{\mathbf{X}}') \cong H^\bullet(\overline{\mathbf{X}})[T]/(\mathfrak{J}T+P(T))$$

with $\mathfrak{J}$ and $P(T)$ as in Theorem II.4.2.

By the Hodge isomorphism theorem [AP21], see Section II.2.7, we have isomorphisms $A^p(\Sigma') \xrightarrow{\sim} H^{p,p}(\overline{X}')$ and $A^p(\Sigma) \xrightarrow{\sim} H^{p,p}(\overline{X})$ for each p. Moreover, since $\mathbf{X}$ is cohomologically tropical by Lemma II.4.1, all the cohomology groups $H^{p,q}(\overline{X}')$ and $H^{p,q}(\overline{X})$ are vanishing for $p \neq q$.

The isomorphism $H^\bullet(\overline{X}') \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}')$ now follows from the commutativity of the diagram in Remark II.3.3, the isomorphisms $H^\bullet(\overline{X}) \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}})$ and $H^\bullet(\overline{X}^\sigma) \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}^\sigma)$, and the compatibility of the decompositions in Keel's lemma in the tropical and algebraic settings with respect to these isomorphisms.

Consider now an arbitrary cone δ of Σ' and denote by η the smallest cone of Σ which contains δ. The star fan Σ'^δ of δ in Σ' is isomorphic to a product of two fans $\Delta \times \Theta$ with Δ a unimodular fan living in $N^\eta_\mathbb{R}$ and Θ a unimodular fan living in $N_{\sigma,\mathbb{R}}/N_{\delta\cap\sigma,\mathbb{R}}$. In the case $\eta \not\preccurlyeq \sigma$, the first fan Δ coincides with the star fan Σ^η of η in Σ. Otherwise, when $\eta \preccurlyeq \sigma$, Δ is the fan obtained from Σ^η by subdividing the cone σ/η. The other fan Θ is $\underline{0}$ unless δ contains the ray ρ in which case, Θ is the fan of the projective space of dimension $|\sigma| - |\sigma \cap \delta|$. Similarly, $\overline{\mathbf{X}}'^\delta$ admits a decomposition into a product $\mathbf{Y} \times \mathbf{Z}$, where $\mathbf{Y} = \overline{\mathbf{X}}^\eta$ in the case $\eta \not\preccurlyeq \sigma$, and $\mathbf{Y}$ is the blow-up of $\overline{\mathbf{X}}^\sigma$ in $\overline{\mathbf{X}}^\eta$ in the other case $\eta \preccurlyeq \sigma$. And $\mathbf{Z}$ is $\mathbb{CP}^0$, that is a point, unless δ contains ρ in which case $\mathbf{Z} \cong \mathbb{CP}^{|\sigma|-|\sigma\cap\delta|}$.

The isomorphism $H^\bullet(\overline{X}'^\delta) \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}'^\delta)$ for δ can be then obtained from the above description, and by observing that when σ is face of η and Δ is the subdivision of η/σ in Σ^η, we can apply the argument used in the first treated case above to $\overline{\mathbf{X}}^\eta$ and $\overline{X}^\eta$ to conclude.

Consider now the case where Σ is obtained as a barycentric star subdivision of Σ'. We only discuss the isomorphism $H^\bullet(\overline{X}') \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}')$, the other isomorphisms $H^\bullet(\overline{X}'^\delta) \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}'^\delta)$ for $\delta \in \Sigma'$ can be obtained by using the preceding discussion. The cohomology of $\overline{X}'$ appears as a summand of the cohomology of $\overline{X}$ according to the decomposition in Keel's lemma. Similarly, the cohomology of $\overline{\mathbf{X}}'$ is a summand of the cohomology of $\overline{\mathbf{X}}$. Using the compatibility of the decompositions in the Keel's lemma, the isomorphism $H^\bullet(\overline{X}) \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}})$ induces an isomorphism $H^\bullet(\overline{X}') \xrightarrow{\sim} H^\bullet(\overline{\mathbf{X}}')$ between the two summands. ∎

Theorem II.4.5. *Suppose that the subvariety $\mathbf{X} \subseteq \mathbf{T}$ is wunderschön with respect to some unimodular fan. Then $\mathbf{X}$ is wunderschön with respect to any unimodular fan with support $X = \mathrm{trop}(\mathbf{X})$.*

Proof. The proof of this theorem is similar to the proof given above for Theorem II.4.4. We omit the details. ∎

II.5 Divisorial cohomology

In this section, we prove Theorem II.5.1 which states that the cohomology of a wunderschön variety is divisorial.

The cohomology of a non-singular algebraic variety $\mathbf{Z}$ is *divisorial* if there is a surjective ring homomorphism $\mathbb{Q}[\mathrm{x}_1, \dots, \mathrm{x}_s] \to H^\bullet(\mathbf{Z})$ such that the image of each x_i is $[\mathbf{D}_i] \in H^2(\mathbf{Z})$, the Poincaré dual of some divisor $\mathbf{D}_i$ of $\mathbf{Z}$. Similarly, the Chow ring $A^\bullet(\mathbf{Z})$ is *divisorial* if there is a surjective ring homomorphism $\mathbb{Q}[\mathrm{x}_1, \dots, \mathrm{x}_s] \to A^\bullet(\mathbf{Z})$ such that the image of each x_i is the class of a divisor $\mathbf{D}_i$ of $\mathbf{Z}$. In this case, we also say that the (Chow) cohomology of $\mathbf{Z}$ is generated by the divisors $\mathbf{D}_1, \dots, \mathbf{D}_s$. Notice that if $\mathbf{Z}$ is projective and its cohomology is divisorial, then all its cohomology is generated by algebraic cycles and the Hodge structure on the cohomology is Hodge-Tate.

The Chow ring of any non-singular complex toric variety is divisorial and generated by the toric boundary divisors, see [Bri96, Section 3.1] and Section II.2.7. It follows, using our previous notations, that if the the map $\tau^*\colon H^\bullet(\overline{X}) \to H^\bullet(\overline{\mathbf{X}})$ is a surjection, then the cohomology of $\overline{\mathbf{X}}$ is divisorial and generated by the irreducible components of $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$.

Theorem II.5.1. *Let $\mathbf{X} \subseteq \mathbf{T}$ be a wunderschön subvariety. Let $\overline{\mathbf{X}}$ be the compactification of $\mathbf{X}$ with respect to a unimodular fan Σ with support $X = \mathrm{trop}(\mathbf{X})$. Then the cohomology of $\overline{\mathbf{X}}$ is divisorial and generated by irreducible components of $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$.*

Proof. We proceed by induction on the dimension of $\mathbf{X}$. If $\mathbf{X}$ is a point, then this is trivial. Notice also that if $\mathbf{X}$ is a wunderschön curve then $\overline{\mathbf{X}}$ must be $\mathbb{CP}^1$ and hence the cohomology is divisorial as $H^\bullet(\mathbb{CP}^1) \cong \mathbb{Q}[\mathrm{x}]/\langle \mathrm{x}^2 \rangle$.

We have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\rho\in\Sigma_1} \mathbb{Q}[\mathrm{x}_\zeta \mid \zeta \in \Sigma_1 \text{ and } (\rho+\zeta)\in\Sigma_2] & \xrightarrow{\bigoplus_\rho f_\rho} & \bigoplus_{\rho\in\Sigma_1} H^\bullet(\overline{\mathbf{X}}^\rho) \\
\big\downarrow{\scriptstyle \bigoplus_\rho -\cdot \mathrm{x}_\rho} & & \big\downarrow{\scriptstyle \mathrm{Gys}} \\
\mathbb{Q}[\mathrm{x}_\zeta \mid \zeta \in \Sigma_1] & \xrightarrow{\quad f \quad} & H^{\bullet+2}(\overline{\mathbf{X}}),
\end{array}$$

where $\rho+\zeta$ is the cone generated by the rays ρ and ζ, the f_ρ are surjective ring homomorphisms which send x_ζ to $[\overline{\mathbf{X}}^{\rho+\zeta}]$, and f maps x_ζ to $[\overline{\mathbf{X}}^\zeta]$. Since $\mathbf{X}$ is wunderschön the maps

$$\bigoplus_{\rho\in\Sigma_1} H^k(\overline{\mathbf{X}}^\rho) \to H^{k+2}(\overline{\mathbf{X}})$$

from the Deligne weight spectral sequence are all surjections for $k \geq 0$ and we deduce that f is surjective. Therefore, the cohomology of $\overline{\mathbf{X}}$ is divisorial and is generated by the components of $\overline{\mathbf{X}} \setminus \mathbf{X}$. ■

II.6 Proof of the main theorem

We now turn to proving Theorem II.6.1.

Theorem II.6.1. *Let* $\mathbf{X} \subseteq \mathbf{T}$ *be a schön subvariety with tropicalization* $X = \mathrm{trop}(\mathbf{X})$. *Then the following statements are equivalent.*

II.6.1.1. $\mathbf{X}$ *is wunderschön and* X *is a tropical homology manifold,*

II.6.1.2. $\mathbf{X}$ *is cohomologically tropical.*

Moreover, if any of these statements holds, then X *is Kähler.*

Proof. We begin by assuming that $\mathbf{X}$ is wunderschön and that X is a tropical homology manifold, and prove that $\mathbf{X}$ is cohomologically tropical. We must show that the maps $\tau^* \colon H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ are isomorphisms for all $\sigma \in \Sigma$. Notice that $\mathbf{X}$ is non-singular since it is wunderschön.

If $\mathbf{X}$ is of dimension 0 and wunderschön it consists of a single point. Therefore, its tropicalization is a point of weight 1 thus $\mathbf{X}$ is cohomologically tropical. We proceed by induction on the dimension of $\mathbf{X}$. Therefore, we can assume that each of the $\mathbf{X}^\sigma$ is cohomologically tropical for all cones $\sigma \in \Sigma$ not equal to the origin.

Since $\mathbf{X}$ is schön, let $\mathbf{D} = \overline{\mathbf{X}} \setminus \mathbf{X}$ be the simple normal crossing divisor of the compactification. The Deligne weight spectral sequence for the tropical compactification $(\overline{\mathbf{X}}, \mathbf{D})$ of $\mathbf{X}$ abuts in the associated graded objects of the weight filtration of the cohomology of $H^k(\mathbf{X})$. Since $\mathbf{X}$ is wunderschön, the E_1-page of Deligne spectral sequence extends to exact rows by Lemma II.2.4, with the morphisms being sums of Gysin maps. In the tropical setting, since X is a tropical homology manifold, there are tropical Deligne resolutions Section II.2.8, where the maps are sums of tropical Gysin maps.

Now by induction, $\tau^* \colon H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ is an isomorphism, and moreover the appropriate commutative diagrams using the classical and tropical Gysin maps commute by Proposition II.3.2. We may therefore identify the two exact sequences. Applying the five lemma in the cases $k \geq 2$, exactness gives us isomorphisms $H^k(X) \to H^k(\mathbf{X})$ and $\tau^* \colon H^{2k}(\overline{X}) \to H^{2k}(\overline{\mathbf{X}})$. For $k = 0$, since $\mathbf{X}$ is assumed to be connected, there is an isomorphism $H^0(\overline{X}) \cong \mathbb{Q} \cong H^0(\overline{\mathbf{X}})$, and it merely remains to show the claim for $k = 1$.

We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X) & \longrightarrow & \bigoplus_{\zeta\in\Sigma_1} H^0(\overline{X}^\zeta) & \longrightarrow & H^2(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \bigoplus \tau^* & & \downarrow \tau^* & & \\ 0 & \longrightarrow & H^1(\mathbf{X}) & \longrightarrow & \bigoplus_{\zeta\in\Sigma_1} H^0(\overline{\mathbf{X}}^\zeta) & \xrightarrow{g} & H^2(\overline{\mathbf{X}}) & \longrightarrow & 0. \end{array}$$

By induction, the middle vertical arrow is an isomorphism, and we wish to show that the rightmost vertical arrow is an isomorphism. By a diagram chase, exactness of the lower row implies that this arrow is surjective. The injectivity follows from Proposition II.3.4. Therefore, the map $\tau^*\colon H^2(\overline{X}) \to H^2(\overline{\mathbf{X}})$ is an isomorphism. Together with our induction assumption on the maps τ^* this proves that $\mathbf{X}$ is cohomologically tropical.

Now assume that $\mathbf{X}$ is cohomologically tropical. By Lemma II.4.1, we know that X is a tropical homology manifold. It remains to show that $\mathbf{X}$ is wunderschön. We again proceed by induction on dimension as the case for points is trivial. We equip $\mathbf{X}$ with the tropical compactification $\overline{\mathbf{X}}$ given by Σ, such that all open $\mathbf{X}^\sigma$ are wunderschön by induction, for σ different from the central vertex of Σ. We have $H^0(\overline{\mathbf{X}}) \cong H^0(\overline{X})$ by hypothesis, and $H^0(\overline{X}) \cong \mathbb{Q}$, thus $\overline{\mathbf{X}}$ is connected and so is $\mathbf{X}$. It remains to show that the mixed Hodge structure on $H^k(\mathbf{X})$ is pure of weight $2k$ for each k. This follows from comparing the Deligne weight spectral sequence and tropical Deligne resolution by Proposition II.3.2, using that all the maps τ^* are isomorphisms. Hence $\overline{\mathbf{X}}$ is wunderschön.

Finally we prove that if $\mathbf{X}$ is cohomologically tropical, then X is Kähler. By Lemma II.4.1, we know that X is a tropical homology manifold. There exists a unimodular fan Σ with support X such that Σ is quasi-projective. It follows that the Chow rings $A^{\bullet/2}(\Sigma^\sigma)$, $\sigma \in \Sigma$, are isomorphic to $H^\bullet(\overline{\mathbf{X}}^\sigma)$. Moreover, since Σ is quasi-projective, and $\mathbf{X}$ is schön, $\overline{\mathbf{X}}^\sigma$ is a non-singular projective variety, and so its cohomology verifies the Kähler package. We conclude that X is Kähler. ∎

Theorem II.6.2 (Isomorphism of cohomology on open strata)**.** *Suppose that* $\mathbf{X} \subseteq \mathbf{T}$ *is schön and cohomologically tropical. Let* Σ *be any unimodular fan with support* $X = \operatorname{trop}(\mathbf{X})$*. Then we obtain isomorphisms*

$$\tau^*\colon H^k(X^\sigma) \xrightarrow{\sim} H^k(\mathbf{X}^\sigma)$$

for all $\sigma \in \Sigma$ *and all* k*.*

Proof. It suffices to prove the statement for $\mathbf{X}$, since if $\mathbf{X}$ is cohomologically tropical so are all strata $\mathbf{X}^\sigma$. It follows from the proof of Theorem II.6.1, that if $\mathbf{X}$ is cohomologically tropical, then $\mathbf{X}$ is wunderschön and hence the $2k$-th row of the 1st page of the Deligne weight spectral sequence provides a resolution of $H^k(\mathbf{X})$ for all k. Moreover, the maps $\tau^*\colon H^\bullet(\overline{X}^\sigma) \to H^\bullet(\overline{\mathbf{X}}^\sigma)$ are isomorphisms for all strata and they commute with the tropical and complex

Gysin maps. Therefore, we obtain an isomorphisms of the resolutions which induces isomorphisms $\tau^*\colon H^k(X^\sigma) \to H^k(\mathbf{X}^\sigma)$ for all $\sigma \in \Sigma$ and all k. ■

II.7 Globalization

We discuss a natural extension of the main theorem of [IKMZ19]. We follow the setting of that work. Let $\pi\colon \mathfrak{X} \to D^*$ be an algebraic family of non-singular complex algebraic varieties in $\mathbb{CP}^n$ over the punctured disk D^*. Let $Z \subseteq \mathbb{TP}^n$ be the tropicalization of the family. We suppose Z admits a unimodular triangulation. This is always possible after a base change of the form $D^* \to D^*$, $z \mapsto z^k$, for $k \in \mathbb{Z}_+$. Using the triangulation, we construct a degeneration of $\mathbb{CP}^n$ to an arrangement of toric varieties, and taking the closure of the family $\mathfrak{X}$ inside this toric degeneration leads to a family $\overline{\mathfrak{X}}$ extended over the full punctured disk D. By Mumford's proof of the semistable reduction theorem, we can always find a triangulation, after a suitable base change, such that the extended family is regular and the fiber over zero is reduced and simple normal crossing. This is known as a semistable extension of the family $\pi\colon \mathfrak{X} \to D^*$.

Denote by $\mathfrak{X}_0$ the fiber at zero of the extended family. Note that since the extended family is obtained by taking the closure of the family in a toric degeneration of $\mathbb{CP}^n$, each open stratum in $\mathfrak{X}_0$ will be naturally embedded in an algebraic torus. For $t \in D^*$ denote by $\mathfrak{X}_t$ the fiber of π over t.

Theorem II.7.1. *Let $\pi\colon \mathfrak{X} \to D^*$ be an algebraic family of subvarieties in $\mathbb{CP}^n$ parameterized by the punctured disk and let $\pi\colon \overline{\mathfrak{X}} \to D$ be a semistable extension. If the tropicalization $Z \subseteq \mathbb{TP}^n$ is a tropical homology manifold and all the open strata in $\mathfrak{X}_0$ are wunderschön, then $H^{p,q}(Z)$ is isomorphic to the associated graded piece W_{2p}/W_{2p-1} of the weight filtration in the limiting mixed Hodge structure $H^{p+q}_{\lim}$. The odd weight graded pieces in $H^{p+q}_{\lim}$ are all vanishing.*

Moreover, for $t \in D^*$, we have $\dim H^{p,q}(\mathfrak{X}_t) = \dim H^{p,q}(Z)$, for all non-negative integers p and q.

Proof. Since Z is a tropical homology manifold, the local fanfolds appearing in the tropical variety Z are all tropical homology manifolds. Moreover, the Chow ring of any unimodular fan supported in a local fanfold of Z is the cohomology ring of a non-singular proper complex algebraic variety. It follows that this Chow ring verifies the Kähler package provided that the fan is quasi-projective. We apply now the Steenbrink-Tropical comparison theorem proved in [IKMZ19; AP20] to obtain the isomorphism between the cohomology groups $H^{p,q}(Z)$ with the cohomology of the Steenbrink sequence in weight $2p$ associated to the triangulation, on one side, and the vanishing in the odd-weight of the cohomology of the Steenbrink sequence on the other side. The Steenbrink spectral sequence gives the weight $2p$ part of the limit mixed Hodge structure in degree $p+q$. The wunderschön assumption implies that the limit mixed Hodge structure is of Hodge-Tate type. We conclude similarly to the proof of Corollary 2 in [IKMZ19]. ■

The following statement shows that degenerations appearing in the above theorem are all maximal.

Theorem II.7.2. *Notations as in Theorem II.7.1, the family $\mathfrak{X} \to D^*$ is maximally degenerate.*

Proof. By the Deligne weight spectral sequence, each closed stratum in $\mathfrak{X}_0$ has a cohomology of Hodge-Tate type. Steenbrink spectral sequence then shows that the limit mixed Hodge structure is Hodge-Tate. ∎

We discuss maximal degenerations further in Section II.8.4.

II.8 Discussions

II.8.1 Examples

In this section, we give various examples of varieties verifying some but not all conditions of the main Theorem II.6.1. These examples tend to demonstrate that the main theorem cannot be weakened.

II.8.1.1 A wunderschön variety which is not cohomologically tropical

Take $N = \mathbb{Z}^2$. Let $\mathbf{X} \subseteq \mathbf{T}_N$ be the conic given by the equation $a + bz_1 + cz_2 + dz_1z_2 = 0$ for generic complex coefficients a, b, c and d. The variety $\mathbf{X}$ is $\mathbb{CP}^1$ with four points removed. This is a wunderschön variety: looking at the compactification $\overline{\mathbf{X}} \subseteq (\mathbb{CP}^1)^2$, $\mathbf{X}$ is non-singular and the intersections with torus orbits are the points hence non-singular, so that $\mathbf{X}$ is schön. Moreover, each of the points removed is trivially wunderschön. Finally, the Deligne weight spectral sequence shows that $\mathbf{X}$ has pure Hodge structure. However, the tropicalization X of $\mathbf{X}$ is the union of the axes in $\mathbb{R}^2$, which is not uniquely balanced,, i.e.,, $\dim H^2(X) = 2 > 1$. This means that X is not a tropical homology manifold (see [Aks23, Theorem 4.8]). Moreover, computing the cohomology groups of $\overline{X}$, we obtain $\dim H^0(\overline{X}) = 1$, $\dim H^1(\overline{X}) = 0$ and $\dim H^2(\overline{X}) = 2$, which differs from the cohomology groups of the sphere $\overline{\mathbf{X}}$.

II.8.1.2 A schön variety with pure strata, whose tropicalization is a tropical homology manifold but which is not cohomologically tropical

Let $\mathbf{X}$ be a generic conic in $(\mathbb{C}^*)^2$. The variety $\mathbf{X}$ is $\mathbb{CP}^1$ with six points removed. Its tropicalization is the usual tropical line equipped with weights equal to 2 on all edges, hence again a tropical homology manifold by [Aks23, Theorem 4.8]. The variety $\mathbf{X}$ is schön since it is non-singular, and each one of the three strata consists of two distinct points, hence it is non-singular. The mixed Hodge structure on $\mathbf{X}$ is pure, as the Deligne weight spectral sequence shows that $\mathrm{Gr}_1^W H^1(\mathbf{X}) = H^1(\overline{\mathbf{X}}) = 0$. However, it is not wunderschön since its strata are not connected. The map $\tau^* \colon H^\bullet(\overline{X}) \to H^\bullet(\overline{\mathbf{X}})$ is an isomorphism: it maps the class of a point in $\overline{X}$ to twice the class of a point in $\overline{\mathbf{X}}$. Nevertheless, $\mathbf{X}$ is not cohomologically tropical since, for any ray ζ of X, $H^0(\overline{\mathbf{X}}^\zeta) \cong \mathbb{Q}^2$ but $H^0(\overline{X}^\zeta) \cong \mathbb{Q}$.

II.8.1.3 A schön variety which is not pure nor cohomologically tropical and whose tropicalization is a tropical homology manifold

Consider the punctured elliptic curve $\mathbf{X}$ in $(\mathbb{C}^*)^2$ of equation $az_1^2+bz_2+cz_1z_2^2 = 0$ for generic complex coefficents a, b and c. Topologically it is a torus punctured in three points. The tropicalization is the unimodular tropical line of weight one with rays generated by $(2, 1), (-1, 1)$ and $(-1, -2)$, which is a tropical homology manifold. The variety $\mathbf{X}$ is non-singular and connected, and each of the three strata at infinity of its compactification is a point hence non-singular and connected. Hence $\mathbf{X}$ is schön. The cohomology group $H^1(\overline{\mathbf{X}})$ is nontrivial of dimension 2. However, $H^1(\overline{X})$ is trivial. Hence $\mathbf{X}$ is not cohomologically tropical. This is because $\mathbf{X}$ is not wunderschön. More precisely, $H^1(\mathbf{X})$ is not pure of weight 2. Indeed, by the Deligne weight spectral sequence $\mathrm{Gr}_1^W(H^1(\mathbf{X})) \cong H^1(\overline{\mathbf{X}}) \neq 0$.

II.8.1.4 A non-schön variety which is cohomologically tropical

Once again, N is of dimension 2. Let $\mathbf{X} \subseteq \mathbf{T}_N$ be given by the equation $(z_1 - a)(z_2 - b) = 0$ for $a, b \neq 0$. The variety $\mathbf{X}$ is a reducible nodal curve with two components both being $\mathbb{CP}^1$ with two punctures. The tropicalization is again the union of the two coordinate axes in $\mathbb{R}^2$, which is not a tropical homology manifold, and the variety $\mathbf{X}$ is not schön as it is singular. However, for each line of the cross, the cocycle associated to this line is mapped to the cocycle associated to the corresponding sphere. This is an isomorphism between $H^2(X)$ and $H^2(\mathbf{X})$. Since $H^0(X)$ is trivially isomorphic to $H^0(\mathbf{X})$ and other cohomology groups are trivial, we deduce that $\mathbf{X}$ is cohomologically tropical.

II.8.2 Hyperplane arrangement complements

We will now see that all three properties of Theorem II.6.1 are satisfied for complements of projective hyperplane arrangements. We will use the de Concini-Procesi model of the complement of a projective hyperplane arrangement [DP95], as discussed in [MS15, Section 4.1]. Let $\mathcal{A} = \{H_i\}_{i=0}^n$ be an arrangement of $n + 1$ hyperplanes in $\mathbb{P}^d_{\mathbb{C}}$, not all having a common intersection point, and let $\mathbf{X}_{\mathcal{A}} = \mathbb{P}^d_{\mathbb{C}} \setminus \bigcup_{H_i \in \mathcal{A}} H_i$ be the complement of the arrangement. For each i, let ℓ_i be the homogeneous linear form such that $H_i = \{z \in \mathbb{P}^d_{\mathbb{C}} \mid \ell_i(z) = 0\}$. These define a map $\mathbf{X}_{\mathcal{A}} \to (\mathbb{C}^*)^n$ given by $z \mapsto (\ell_i(z))$ in homogeneous coordinates on $(\mathbb{C}^*)^n$. This map is injective, since no $z \in \mathbf{X}_{\mathcal{A}}$ lies on all hyperplanes by assumption, and induces an isomorphism of $\mathbf{X}_{\mathcal{A}} \cong \mathbf{Y}_{\mathcal{A}}$, where $\mathbf{Y}_{\mathcal{A}}$ is a subvariety of $(\mathbb{C}^*)^n$, see [MS15, Proposition 4.1.1] for details. By a theorem of Ardila and Klivans [AK06], the tropicalization $Y_{\mathcal{A}} = \mathrm{Trop}(\mathbf{Y}_{\mathcal{A}})$ is the support of the

Bergman fan $\Sigma_{M_\mathcal{A}}$ of the matroid $M_\mathcal{A}$ associated to the arrangement $\mathcal{A}$, see [MS15, Sections 4.1–4.2].

First, Tevelev shows [Tev07, Theorem 1.5] that the variety $\mathbf{X}_\mathcal{A}$ is schön, it is clearly connected, and by [Sha93], its cohomology has a pure Hodge structure of Hodge-Tate type. Moreover, given a face $\sigma \in \Sigma_{M_\mathcal{A}}$, the star fan of σ corresponds to the complement of a hyperplane arrangement. By induction, this shows that complements of hyperplane arrangements are wunderschön.

Furthermore, it is shown in [JRS18; JSS19] by an inductive argument that the Bergman fan $\Sigma_{M_\mathcal{A}}$ of a matroid is a tropical homology manifold. Therefore, one can apply Theorem II.6.1, which gives us that $\mathbf{Y}_\mathcal{A}$ is cohomologically tropical, i.e. the map $\tau^*\colon H^\bullet(\overline{Y}_\mathcal{A}) \to H^\bullet(\overline{\mathbf{Y}}_\mathcal{A})$ is an isomorphism.

In light of Theorem II.6.2, this can be compared with the main result of [Zha13], also independently proved in [Sha11], showing that $H^\bullet(\mathbf{X}_\mathcal{A}) \cong H^\bullet(X_\mathcal{A})$, however lacking explicit maps.

II.8.3 A non-matroidal example

We present an example of $\mathbf{X} \subseteq \mathbf{T}_N$ which is not a complement of a hyperplane arrangement yet is wunderschön, cohomologically tropical, and the tropicalization $\operatorname{trop}(\mathbf{X})$ is a tropical homology manifold.

The variety $\mathbf{X}$ will be the complement of an arrangement of lines and a single conic in $\mathbb{CP}^2$. Let $[z_0 : z_1 : z_2]$ be homogeneous coordinates on $\mathbb{CP}^2$. Let $\mathbf{L}_0$, $\mathbf{L}_1$, and $\mathbf{L}_2$ be the coordinate lines of $\mathbb{CP}^2$ so that $\mathbf{L}_i$ is defined by $z_i = 0$. Let $\mathbf{L}_3$ be defined by the linear form $z_0 - z_1 + z_2 = 0$ and let the conic $\mathbf{C}$ be defined by $z_1^2 + z_2^2 - z_0 z_1 - 2z_1 z_2 = 0$. Let $\mathcal{A}$ denote the union of $\mathbf{L}_0, \dots, \mathbf{L}_3, \mathbf{C}$.

As depicted in Figure II.3, note that $\mathbf{C}$ is tangent to $\mathbf{L}_1$ at the point $[1 : 0 : 0]$ where $\mathbf{L}_1$ intersects $\mathbf{L}_2$. Also the conic $\mathbf{C}$ is tangent to $\mathbf{L}_0$ at the intersection point $[0 : 1 : 1]$ with $\mathbf{L}_3$. The conic also passes through the intersection point $[1 : 1 : 0]$ of $\mathbf{L}_2$ and $\mathbf{L}_3$.

Consider the map $\phi\colon \mathbb{CP}^2 \setminus \mathcal{A} \to (\mathbb{C}^*)^4$ defined by

$$[z_0 : z_1 : z_2] \mapsto (\widetilde{z}_1, \widetilde{z}_2, 1-\widetilde{z}_1+\widetilde{z}_2, \widetilde{z}_1^2+\widetilde{z}_2^2-\widetilde{z}_1-2\widetilde{z}_1\widetilde{z}_2), \quad \text{with } \widetilde{z}_1 = \frac{z_1}{z_0} \text{ and } \widetilde{z}_2 = \frac{z_2}{z_0}.$$

Let $\mathbf{X} \subseteq (\mathbb{C}^*)^4$ denote the image of the map ϕ. The space $\operatorname{trop}(\mathbf{X})$ is 2-dimensional and is the support of the fan described below.

The fan has 8 rays in directions given in Figure II.4. Each ray is adjacent to exactly 3 faces of dimension 2 for a total of 12 faces of dimension 2. The structure is given in Figure II.4: we draw an edge between two vertices if the there is a face between the two corresponding rays. Note that to get a unimodular subdivision, one has to add some rays, for instance the rays α and β of Figure II.4. We denote by Σ this unimodular fan.

It can be verified in polymake that this fan is a tropical homology manifold and its tropical Betti numbers are $1, 0, 6, 0, 1$. For an alternative proof, note that the the fan Σ is obtained by the process of tropical modification [MR18] as follows. Let $\Sigma_{U_{3,4}} \subseteq \mathbb{R}^3$ be the Bergman fan of the uniform matroid $U_{3,4}$. Its rays are the rays $0, 1, 2$ and 3 in Figure II.4, where we forget the fourth

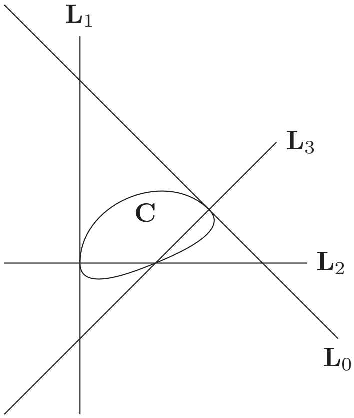

Figure II.3: A very-affine variety $\mathbf{X}$ which is not a complement of hyperplane arrangement and which verifies the main theorem, see Section II.8.3.

0	1	2	3	4	a	b	c
-1	1	0	0	0	2	0	-2
-1	0	1	0	0	1	1	-2
-1	0	0	1	0	0	1	-1
-2	0	0	0	1	2	1	-2

α	β
1	-1
1	-1
0	0
0	-1

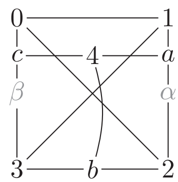

Figure II.4: The combinatorial structure of a non-Bergman fan verifying the main theorem described in Section II.8.3.

coordinate. Let $C \subseteq \Sigma_{U_{3,4}}$ be a tropical trivalent curve with rays a, b, c (once again we forget the last coordinate). Then Σ in $\mathbb{R}^4$ is obtained by a tropical modification of $\Sigma_{U_{3,4}}$ along C. By [JRS18], the Bergman fan $\Sigma_{U_{3,4}}$ is a tropical homology manifold, see also Section II.8.2. By [Aks23, Theorem 4.8] the trivalent tropical curve is also a tropical homology manifold. By [AP21] the modification of $\Sigma_{U_{3,4}} \subseteq \mathbb{R}^3$ along C is a tropical homology manifold. The tools developed in this last article also allow to compute the cohomology of $\overline{X}$ quite easily, and to check that the fan is Kähler.

The compactification of $\mathbf{X}$ in $\mathbb{CP}_\Sigma$ is given as follows. Consider $\mathbb{CP}^2$ blown up in the three points whose homogeneous coordinates are $[1:0:0], [0:1:1]$, and $[1:1:0]$. Then, in the blow up, the exceptional divisor above $[1:0:0]$, the proper transform of $\mathbf{C}$, and the proper transform of $\mathbf{L}_1$ all intersect in a single point. Similarly, there is a triple intersection of the exceptional divisor above $[0:1:1]$ and the proper transforms of $\mathbf{C}$ and $\mathbf{L}_0$. We further blow-up these two intersection points to obtain a surface $\overline{\mathbf{X}}$. The divisor $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$ consists of the five exceptional divisors and the proper transforms of all curves in $\mathcal{A}$. Therefore, $\dim H^2(\overline{\mathbf{X}}) = 6$ and $\dim H^0(\overline{\mathbf{X}}) = \dim H^4(\overline{\mathbf{X}}) = 1$ and $\dim H^k(\overline{\mathbf{X}}) = 0$ otherwise.

We claim that $\mathbf{X}$ is wunderschön. Indeed, for each ray ζ of the fan Σ

the variety $\overline{\mathbf{X}}^{\varsigma}$ is $\mathbb{CP}^1$ with two or three marked points corresponding to the intersections with the other divisors in $\overline{\mathbf{X}} \smallsetminus \mathbf{X}$, so it is wunderschön. Moreover, $\mathbf{X}$ is non-singular and connected, and its cohomology is pure. Hence, $\mathbf{X}$ is wunderschön.

II.8.4 Maximal degenerations

Motivated by the work of Deligne and our results Theorem II.7.1 and Theorem II.7.2, we can ask the following question.

Question II.8.1. *Is there a tropical geometric characterization of maximally degenerate families of complex algebraic varieties? It is true that those families in which the open strata of special fibers have a cohomology which is pure of Hodge-Tate type are exactly those covered by our Theorem II.7.1?*

The question is intimately related to the work of Yang Li [Li20] which reduces the SYZ conjecture in maximally degenerate families of complex algebraic varieties to the existence of solutions to a tropical Monge-Ampère equation (once this has been properly formulated). For those degenerations appearing in Theorem II.7.1, our results show that the corresponding tropical variety is Kähler in the sense of [AP20] and moreover recovers the geometry of the degenerate fiber as well as the limit Hodge-theoretic geometry of the family. Tropical Hodge theory [AP20] can be then used to properly formulate the Monge-Ampère equation on the tropicalization using tropical Kähler forms.

II.8.5 Shellability

It seems plausible that a framework parallel to the one in [AP21; AP23] can be developed for properties of tropicalization of algebraic varieties. The properties discussed in this paper concern pairs consisting of a subvariety of an algebraic torus and a fan structure on its tropicalization. Three basic operations can be conducted on these pairs: products, blow-ups and blow-downs, and taking the graph of a holomorphic function on the subvariety, restricted to the complement of its divisor. For example the cases described in Sections II.8.2 and II.8.3 can both be obtained by these operations. The properties of being schön, wunderschön, and cohomologically tropical should be shellable in this framework. We refer to [Sch21] for some results in this direction.

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Paper III

Cohomologically tropical arroids, curve arrangements and maximality

Edvard Aksnes

Abstract

We define *arroids* as an abstract axiom set encoding the intersection properties of arrangements of curves. The tropicalization of the complement of arrangement of curves meeting pairwise transversely is shown to be determined by the associated arroid. We give conditions for when the cohomology of the complement of an arrangement is computable using tropical cohomology, and we give criteria for when the complement is a maximal variety in terms of tropical geometry.

Contents

III.1 Introduction

Drawing inspiration from matroids, which abstractly axiomatize arrangements of hyperplanes, we define *arroids*, which provide a possible abstract axiom set for the incidence geometry of arrangements of curves in the plane. To any arrangement of curves, one may associate an arroid. An arroid $\mathbf{A}$ consist of an underlying set $\mathcal{A}$ where each element i is equipped with a degree d_i, along with a multiset $\mathcal{P}$ of subsets of $\mathcal{A}$. Each set $\mathbf{p} \in \mathcal{P}$ is equipped with a multiplicity function $m_{\mathbf{p}} \colon \mathbf{p}^2 \to \mathbb{Z}$, and the multiset $\mathcal{P}$ must satisfy a Bézout condition in terms of the multiplicity functions.

When all the multiplicity functions of an arroid are constant taking value one, the arroid is said to be *transversal.* We construct a fan associated to each transversal arroid, and the following theorem shows that such a fan is a tropical variety, i.e. satisfies the balancing condition of tropical geometry, see e.g. [BIMS15; MS15] for definitions.

Theorem III.4.6. *For each transversal arroid* $\mathbf{A}$*, there is a fan* $\Sigma_{\mathbf{A}}$*, called the* fan of $\mathbf{A}$*, which is a balanced tropical variety.*

Using transversal arroids, we proceed to study the tropicalization of the complements of certain types of arrangements of curves. An arrangement of curves is *very affine* if it contains at least three lines intersecting generically, and *transverse* if all curves of the arrangement intersect pairwise transversely. In Section III.4.3, we show that for a transverse very affine arrangement of curves, the tropicalization of the complement is computed by the arroid fan.

Theorem III.4.10. *Let* $\mathcal{B}$ *be a transverse very affine arrangement of curves in the plane* $\mathbb{P}^2_K$*. Then the tropicalization* $\operatorname{trop}(X_{\mathcal{B}})$ *of the complement is supported on the fan of the associated transversal arroid* $\mathbf{A}_{\mathcal{B}}$*.*

For arrangements of lines, Theorem III.4.10 recovers that the tropicalization of the complement is computed using the rank three matroid of the arrangement (see e.g. [MS15, Theorem 4.1.11]), using the Ardila–Klivans fan structure [AK06]. The difficulty in generalizing beyond the transverse case lies primarily in understanding the resolution of singularities that arise when higher order intersections are allowed in the arrangement, as was pointed out in [Cue12, p. 20].

Next, we turn to relating the cohomology of the complement of a very affine transverse arrangement of curves to the tropical cohomology of the fan of its associated transversal arroid. For a reminder on tropical cohomology, see Section III.2.4. For line arrangements and their corresponding matroids, an isomorphism between the matroid Orlik–Solomon algebra [OS80], computing cohomology of the complement using only the intersection properties recorded by the matroid, and tropical cohomology of the matroid fan, was described by Zharkov [Zha13]. We consider transverse very affine arrangements $\mathcal{B}$ of non-singular rational curves in $\mathbb{P}^2_{\mathbb{C}}$, i.e. of lines and conics, such that that no intersection point of the arrangement contains exactly the same curves. Such an arrangement will be called *simple.*

In Proposition III.6.1, we show that the complement of a simple arrangement is *wunderschön* in the sense of [AAPS23, Definition 1.2], which is in this context primarily a restriction on its mixed Hodge structure. In light of [AAPS23, Theorem 6.1], this implies that the complement of simple arrangements are *cohomologically tropical* i.e. its rational cohomology can be computed using the $\mathbb{Q}$-coefficient tropical cohomology of its tropicalization, if and only if the corresponding arroid fan is a tropical homology manifold.

Theorem III.6.2. *Let $X_{\mathcal{B}}$ be the complement of a simple arrangement $\mathcal{B}$. Then $X_{\mathcal{B}}$ is cohomologically tropical if and only if the corresponding arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$ is uniquely balanced along each of its rays.*

This result follows from equivalent conditions for an arroid fan to be a tropical homology manifold given in Theorem III.5.4, and we study which conditions this imposes on curve arrangements in Section III.5.2.

Using Theorem III.6.2, we study the question of *maximality* for a real arrangement and its complexification. Let X be a complex variety defined over $\mathbb{R}$, with $X(\mathbb{R})$ its set of real points and $X(\mathbb{C})$ its set of complex points. The *Smith-Thom inequality* gives bounds for the sum of the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers as follows,

$$b_\bullet(X(\mathbb{R})) := \sum_{i\geq 0} b_i(X(\mathbb{R})) \leq \sum_{i\geq 0} b_i(X(\mathbb{C})) =: b_\bullet(X(\mathbb{C})),$$

and the variety is *maximal* if equality is achieved. In [BS22], varieties with torsion-free cohomology satisfying the stronger inequalities

$$b_i(X(\mathbb{R})) \leq \sum_j h^{i,j}(X(\mathbb{C}))$$

in terms of the Hodge numbers of their complex parts are called *sub-Hodge expressive*. In [RS23], Renaudineau and Shaw studied real algebraic hypersurfaces near the tropical limit, and gave bounds for Betti numbers in terms of tropical homology. Recently, [AM22] study the central fiber of a totally real semistable degeneration over a curve. They give three conditions on the components of the central fiber for each of the nearby fibers to be sub-Hodge expressive. For each open component X of the central fiber, the conditions are the following:

(a) $H^i(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i \geq 1$,

(b) X is a maximal variety, and

(c) the mixed Hodge structure on $H^i(X(\mathbb{C}); \mathbb{Q})$ is pure of type (i, i) and integer cohomology is torsion free, for all degrees i.

In light of these conditions, in Theorem III.7.3 we give conditions for the complement of a simple arrangement of real curves in the plane to be maximal using its tropicalization. Moreover, using the wunderschön and cohomologically tropical properties, we give the following concrete construction of varieties satisfying the above conditions.

Theorem III.7.5. *Let $\mathcal{B}$ be a simple arrangement of real curves in $\mathbb{P}^2_{\mathbb{C}}$, with all intersection points being real, and such that the tropicalization* $\mathrm{Trop}(X_{\mathcal{B}})$*, which is supported on the arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$, is a tropical homology manifold. Then the following four properties are satisfied:*

(a) $H^i(X_{\mathcal{B}}(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) = 0$ *for* $i \geq 1$,

(b) $X_{\mathcal{B}}$ *is a maximal variety,*

(c) *the mixed Hodge structure on* $H^i(X_{\mathcal{B}}(\mathbb{C});\mathbb{Q})$ *is pure of type* (i,i) *and* $H^i(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z})$ *is torsion-free for* $i \geq 1$*, and*

(d) $\dim_{\mathbb{Q}} H^i(X_{\mathcal{B}}(\mathbb{C});\mathbb{Q}) = \sum_j \dim_{\mathbb{Q}} H^{i,j}(\Sigma_{\mathbf{A}_{\mathcal{B}}})$ *for each* $i \geq 0$.

This theorem is illustrated by providing an infinite family of maximal surfaces in Example III.7.4, which give examples of the types of variety required in [AM22], using conditions (a), (b) and (c). Condition (d) can be compared to the bounds for Betti numbers given in [RS23].

The paper is structured as follows. In Section III.2, we recall the notions of geometric tropicalization, tropical modifications, and tropical cohomology. In Section III.3, we illustrate how geometric tropicalization can be used to compute the tropicalization of the complement of an arrangement of curves in the plane. In Section III.4, we introduce arroids as an abstract generalization of arrangements of curves, define fans associated to arroids which are *transversal* (see Definition III.4.5), and relate the tropicalization of the complement of an arrangement to the fan of the associated arroid in the transversal case. Next, in Section III.5, we study which arroid fans are tropical homology manifolds, and give some conditions for the arroid fan of an arrangement of lines and conics to be a tropical homology manifold. Finally, in Section III.7, use the concepts developed in the rest of the paper to study the maximality of arrangements of lines and conics.

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III.2 Preliminaries

In this section, we recall certain notions in toric and tropical geometry. For the remainder of this paper, let K be a trivially valued algebraically closed field.

III.2.1 The intrinsic torus

We recall the definition and properties of the intrinsic torus, following [Tev07; ST08; HKT09]. For X a variety, let $\mathcal{O}(X)$ be its coordinate ring, with group of units $\mathcal{O}(X)^*$. The group $\mathcal{O}(X)^*/K^*$ is free abelian of finite rank, and the torus $\mathbf{T}_X \coloneqq \mathrm{Hom}(\mathcal{O}(X)^*/K^*, K^*)$ is called the *intrinsic torus* of X, with lattice of characters $M = \mathcal{O}(X)^*/K^*$. Choosing a splitting of the quotient sequence for M, or equivalently a set of generators, defines a map $X \to \mathbf{T}_X$, and we say that X is *very affine* is this gives a closed embedding. Furthermore, any closed embedding of a variety into an algebraic torus factors through an embedding to the intrinsic torus, composed with a monomial map. See [MS15, Section 6.4] for details.

III.2.2 Geometric tropicalization

We briefly recall the *geometric tropicalization* approach to tropicalization. Let $\mathbf{T}$ be an algebraic torus over the field K, with M its lattice of characters and $N \coloneqq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual lattice. Let $X \subset \mathbf{T}$ be a subvariety and X' a normal, $\mathbb{Q}$-factorial variety birational to X. Any divisor D on X' induces a valuation $\mathrm{val}_D \colon K(X) \to \mathbb{Z}$ on the field of fractions $K(X)$ of X. Such a valuation is called a *divisorial valuation* on $K(X)$, and it defines a vector $[\mathrm{val}_D] \in N_{\mathbb{Q}}$ by restricting a character $m \in M$ to a rational function $m|_X \in K(X)$ and evaluating it using val_D.

The following proposition, originally shown in [HKT09, p. 176], with an alternate proof given in [ST08, Theorem 2.4], characterizes the tropicalization of X using divisorial valuations.

Proposition III.2.1. *The tropicalization* $\mathrm{Trop}(X)$ *is equal to the closure of the subset*

$$\{c[\mathrm{val}_D] \mid c \in \mathbb{R}_{\geq 0},\ \mathrm{val}_D \ \textit{ a divisorial valuation on } K(X)\} \subseteq N_{\mathbb{R}}.$$

For any compactification $\overline{X}$ of $X \subseteq \mathbf{T}$ by a simple normal crossing divisor $\partial X \coloneqq \overline{X} \smallsetminus X$ with irreducible components $D_1, \dots, D_m$, Hacking, Keel and Tevelev give an explicit construction of $\mathrm{Trop}(X)$ as follows (see [HKT09, Theorem 2.3]).

Let $\Delta_{\partial X}$ be the *boundary complex* of $(\overline{X}, \partial X)$, i.e. a simplicial complex with vertices $\{1, \dots, m\}$ containing the simplex $\{i_1, \dots, i_k\}$ if and only if $D_{i_1} \cap \cdots \cap D_{i_k}$ is non-empty. The m divisorial valuations $[\mathrm{val}_{D_1}], \dots, [\mathrm{val}_{D_m}]$ give rays in $N_{\mathbb{R}}$, and for each simplex σ of Δ we form the cone $[\sigma] \coloneqq \mathrm{cone}([\mathrm{val}_{D_i}] \mid i \in \sigma)$ in $N_{\mathbb{R}}$.

Theorem III.2.2 ([HKT09, Theorem 2.3])**.** *For* $X \subseteq \mathbf{T}$ *compactified to* $\overline{X}$ *by a simple normal crossing divisor* D*, we have*

$$\mathrm{Trop}(X) = \cup_{\sigma \in \Delta} [\sigma].$$

Under the assumption of working over a field of characteristic zero, the condition that the divisor D is simple normal crossing can be weakened to being merely *combinatorial normal crossing*, i.e. such that the intersection of any l irreducible components is of codimension l.

Theorem III.2.3 ([Cue12, Theorem 2.8])**.** *For* K *a field of characteristic zero and* $X \subseteq \mathbf{T}$ *compactified to* $\overline{X}$ *by a combinatorial normal crossing divisor* D *we have*

$$\mathrm{Trop}(X) = \cup_{\sigma \in \Delta} [\sigma].$$

Remark III.2.4. The varieties considered in the present paper are surfaces, which admit resolutions of singularities (see [Stacks, Tag 0BIC]), and as such the proofs given in [Cue12, Theorem 2.8] carry through even without assuming that the characteristic of the field K is zero.

III.2.3 Tropical modifications

We recall generalities on tropical divisors and tropical modifications, see [MR18] and [AR10] for details. Let Δ' be a tropical fan of dimension d in $\mathbb{R}^n$ with weight function ω, i.e. satisfying the balancing condition [BIMS15, Definition 5.7]. A *tropical rational function* $\phi\colon \Delta \to \mathbb{R}$ is a function so that Δ' can be subdivided into Δ such that, for each cone σ of Δ, the restricted function $\phi\mid_\sigma$ is integer affine, i.e. there is some $b_\sigma \in \mathbb{R}$ and integer matrix $A_\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^{1\times n}$ such that $\phi\mid_\sigma(\mathbf{x}) = A_\sigma \cdot \mathbf{x} + b_\sigma$.

For each cone γ of Δ, let $L_\mathbb{Z}(\gamma) \subseteq N$ be the (saturated) lattice parallel to σ. Each facet σ of Δ is equipped with a weight $\omega(\sigma)$. Since Δ is tropical, for each codimension one cone τ, and each facet σ containing τ (using notation $\tau \prec \sigma$) one may find a vector $v_{\sigma/\tau} \in L(\sigma)$ such that $L_\mathbb{Z}(\sigma) = L_\mathbb{Z}(\tau) + \mathbb{Z}v_{\sigma/\tau}$ and moreover $\sum_{\tau\prec\sigma} \omega(\sigma)v_{\sigma/\tau} = 0$. Moreover, one may use ϕ to define a weight $\omega_\phi(\tau) \coloneqq \sum_{\tau\prec\sigma} \omega(\sigma)A_\sigma \cdot v_{\sigma/\tau}$. The *divisor* $\operatorname{tropdiv}(\phi)$ *of* ϕ *on* Δ is the weighted polyhedral complex whose maximal cones are the codimension one cones τ such that $\omega_\phi(\tau) \neq 0$.

The *tropical modification* $\mathrm{TM}(\Delta, \operatorname{tropdiv}(\phi))$ of Δ along the tropical rational function ϕ is a fan in $\mathbb{R}^{n+1}$, with cones that we now describe. For each cone σ of Δ, let $\widetilde{\sigma} \coloneqq \{(\mathbf{x}, \phi(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \sigma\}$, and define a weight $\widetilde{\omega}(\widetilde{\sigma}) = \omega(\sigma)$. For each cone τ of $\operatorname{tropdiv}(\phi)$, let $\tau_\geq = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} \mid x \in \tau,\ y \geq \phi(\mathbf{x})\}$, with weight $\widetilde{\omega}(\tau_\geq) = \omega_\phi(\tau)$. The underlying set of $\mathrm{TM}(\Delta, \operatorname{tropdiv}(\phi))$ is the union

$$\mathrm{TM}(\Delta, \operatorname{tropdiv}(\phi)) = (\cup_{\sigma\in\Delta}\widetilde{\sigma}) \cup (\cup_{\tau\in\operatorname{tropdiv}(\phi)}\tau_\geq).$$

Using the weights defined above, $\mathrm{TM}(\Delta, \operatorname{tropdiv}(\phi))$ a balanced polyhedral complex.

The *closed tropical modification* $\mathrm{CTM}(\Delta, \operatorname{tropdiv}(\phi))$ of Δ along the tropical rational function ϕ is a polyhedral complex in $\mathbb{R}^n \times (\mathbb{R} \cup \{\infty\})$, with a cone $\widetilde{\sigma}$ for each $\sigma \in \Delta$, a polyhedron $\tau_\geq$ for each cone τ of $\operatorname{tropdiv}(\phi)$, as well as polyhedra at infinity defined as follows. For each cone τ of $\operatorname{tropdiv}(\phi)$, the corresponding face at infinity τ_∞ is the polyhedron $\tau_\infty = \{(\mathbf{x}, \infty) \in \mathbb{R}^n \times (\mathbb{R} \cup \{\infty\}) \mid x \in \tau\}$. The closed tropical modification $\mathrm{CTM}(\Delta, \operatorname{tropdiv}(\phi))$ is the polyhedrally-decomposed set

$$\mathrm{CTM}(\Delta, \operatorname{tropdiv}(\phi)) = (\cup_{\sigma\in\Delta}\widetilde{\sigma}) \cup (\cup_{\tau\in\operatorname{tropdiv}(\phi)}\tau_\geq) \cup (\cup_{\tau\in\operatorname{tropdiv}(\phi)}\tau_\infty),$$

where again the weights make $\mathrm{CTM}(\Delta, \operatorname{tropdiv}(\phi))$ a balanced polyhedral complex.

III.2.4 Tropical (co)homology

Tropical homology and cohomology, as introduced in [IKMZ19], is an invariant of a rational polyhedral space. While there are several different approaches, see for instance [JRS18; JSS19; AP20; AP21; Aks23; AAPS23; GS23], here we briefly recall the definition of tropical (co)homology as most conveniently defined in the case of fans.

Let Σ be a d-dimensional fan in N, with unique minimal 0-dimensional cone denoted $\mathbf{0}$. Recall from Section III.2.3 that $L_{\mathbb{Z}}(\sigma) \subseteq N$ is the (saturated) lattice parallel to σ. For each $p = 1, \ldots, d$, the *p-th multi-tangent space* is the vector subspace

$$\mathcal{F}_p(\sigma) := \sum_{\sigma \prec \gamma} \bigwedge^p L_{\mathbb{Z}}(\gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq \bigwedge^p N \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the sum is taken over all cones γ of Σ containing σ as a face. Moreover, for τ a face of the cone σ, there is an inclusion $\iota_{\tau \prec \sigma} \colon \mathcal{F}_p(\sigma) \to \mathcal{F}_p(\tau)$ as vector subspaces.

For a general polyhedral complex Δ, the *tropical cohomology* groups $H^{p,q}(\Delta)$ are bigraded vector spaces associated to the complex Δ, however when considering a fan Σ, only the groups $H^{p,0}(\Sigma)$ are non-zero, and given by

$$H^{p,0}(\Sigma) = \mathcal{F}^p(\mathbf{0}) := \operatorname{Hom}_{\mathbb{Q}}(\mathcal{F}_p(\mathbf{0}), \mathbb{Q}),$$

where $\mathbf{0}$ is the unique minimal 0-dimensional cone of Σ. Similarly, the *tropical homology* groups $H_{p,q}(\Sigma)$ of a fan are zero unless $q = 0$, for which we have $H_{p,0}(\Sigma) = \mathcal{F}_p(\mathbf{0})$.

Equip each cone σ with an orientation, and for τ a face of σ, let $\operatorname{sign}(\tau, \sigma)$ be 1 if the chosen orientations of τ and σ are compatible, and -1 otherwise. For each $p = 1, \ldots, d$, we may define the *p-th tropical Borel–Moore complex* $C^{BM}_{p,\bullet}(\Sigma)$ as follows

$$0 \to \oplus_{\alpha \in \Sigma_d} \mathcal{F}_p(\alpha) \xrightarrow{\partial_d} \oplus_{\beta \in \Sigma_{d-1}} \mathcal{F}_p(\beta) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \oplus_{\rho \in \Sigma_1} \mathcal{F}_p(\rho) \xrightarrow{\partial_1} \mathcal{F}_p(\mathbf{0}) \to 0.$$

The differential ∂_k is defined as the sum of its components $(\partial_k)_{\gamma,\delta} \colon \mathcal{F}_p(\gamma) \to \mathcal{F}_p(\delta)$, which is given by $\operatorname{sign}(\delta, \gamma) \cdot \iota_{\delta \prec \gamma}$ if δ is a face of γ, and 0 otherwise. The *tropical Borel–Moore homology* group $H^{BM}_{p,q}(\Sigma)$ is the q-th homology group of the complex $C^{BM}_{p,\bullet}(\Sigma)$.

For any tropical fan Σ of dimension d, the weight function ω gives rise to a *fundamental class* $[\Sigma, \omega] \in H^{BM}_{d,d}(\Sigma)$, and there are cap product maps $\frown [\Sigma, \omega] \colon H^{p,q}(\Sigma) \to H^{BM}_{d-p,d-q}(\Sigma)$. When these maps are isomorphisms, Σ is said to satisfy *tropical Poincaré duality.* If each reduced star Σ^γ of Σ at a cone γ satisfies tropical Poincaré duality, including $\Sigma^{\mathbf{0}} = \Sigma$, then Σ is called a *tropical homology manifold.* Recall that *reduced star* Σ^γ of Σ at a cone γ is the fan in $N/L_{\mathbb{Z}}(\gamma)$ consisting of the projection of the cones δ of Σ containing γ. Note that this is sometimes called simply the *star* in the context of toric geometry, see e.g. [Ful93, Section 3.1], we make the distinction for compatibility with [Aks23].

III.3 Tropicalizing complements of arrangements of curves

We now turn to arrangements of curves $\mathcal{B} := \{L_0, L_1, L_2, C_1, \ldots, C_m\}$ in the plane $\mathbb{P}^2_K$, all distinct and irreducible, with equations $C_k \colon f_k(\mathbf{x}) = 0$, where we include the coordinate axes $L_i \colon x_i = 0$ for $i = 0, 1, 2$. The *complement of the arrangement in the torus* $X := \mathbb{P}^2_K \setminus \cup_{D \in \mathcal{B}} D$ is a very affine variety, and it follows from e.g. [HKT09, Lemma 6.1] that its intrinsic torus $\mathbf{T}_X$ is isomorphic to $(K^*)^{m+2}$ generated by the f_i and coordinates x_1, x_2 for $(K^*)^2 \subset \mathbb{P}^2_K$. We will consider the tropicalization of X as a subvariety of its intrinsic torus.

In [Cue12, Section 5] and [ST08, Proposition 5.3], the following algorithm for computing the tropicalization of X using geometric tropicalization is presented. One iteratively blows up the intersection points of curves in the arrangement until one obtains a combinatorial normal crossing divisor, from which the cones of the tropicalization are extracted. More precisely, one proceeds as follows:

0. Let $\mathcal{B}$ be an arrangement of curves on a surface $\overline{X}$, and let $X := \overline{X} \smallsetminus \cup_{D\in\mathcal{B}} D$. Proceed to (1).

1. If there is a point $\mathbf{p} \in \overline{X}$ such that $\mathbf{p}$ is contained in at least three of the curves of $\mathcal{B}$, proceed to (2), otherwise proceed to (3).

2. Blow up the point $\mathbf{p}$, replace $\overline{X}$ by the blown up surface $\overline{X}'$, and replace the arrangement $\mathcal{B}$ with the arrangement $\mathcal{B}' := \{D'\}_{D\in\mathcal{B}} \cup \{E\}$ of curves on $\overline{X}'$, where E is the exceptional divisor of the blow up, and D' is the strict transform of D. Note that $X' := \overline{X}' \smallsetminus \cup_{C\in\mathcal{B}'} C$ is isomorphic to $X \subset \overline{X}$, and so replace X with X'. Return to (1).

3. Since no three curves of $\mathcal{B}$ have a common intersection point, we may view $\mathcal{B}$ as a combinatorial normal crossing divisor compactifying X into the surface $\overline{X}$.

Using the above procedure, by Theorem III.2.3 the tropicalization $\operatorname{trop}(X)$ can be equipped with a fan structure consisting of one ray for each element D of $\mathcal{B}$, and one two-dimensional cone between the rays for D and D' if these two curves intersect in S. Moreover note that one may ignore the restriction on the characteristic of K by Remark III.2.4. Therefore, to complete the description of the tropicalization $\operatorname{trop}(X)$, it remains to find the directions of these rays.

By [HKT09, p. 182], there is a short exact sequence of abelian groups

$$0 \to \operatorname{Pic}(\overline{X})^\vee \to \bigoplus_{D_i\in\partial X} \mathbb{Z}\cdot D_i \to (\mathcal{O}(X)^*/K^*)^\vee \to 0, \tag{III.1}$$

where the first map is dual to the map $D_i \mapsto [D_i] \in \operatorname{Pic}(\overline{X})$, and the second is the dual of the map defined on generators by $f_i \mapsto \operatorname{div}(f_i) = D_i$. This allows us to compute the directions of the rays associated to the divisors solely in terms of the intersection properties of the curves D in $\mathcal{B}$. We illustrate this with the following examples.

Example III.3.1. Consider the arrangement of curves in $\mathbb{P}^2_K$ consisting of four lines given by equations $L_1\colon x = 0$, $L_2\colon y = 0$, $L_3\colon z = 0$, and $L_4\colon x + z - y = 0$, displayed in Figure III.1. The complement $X := \mathbb{P}^2_K \smallsetminus \cup_i L_i$ of this arrangement is compactified to $\mathbb{P}^2_K$ by the divisor $\partial X = \cup_i L_i$, which is already a simple normal crossing divisor, with boundary complex $\Delta_{\partial X}$ given by the complete graph K_4.

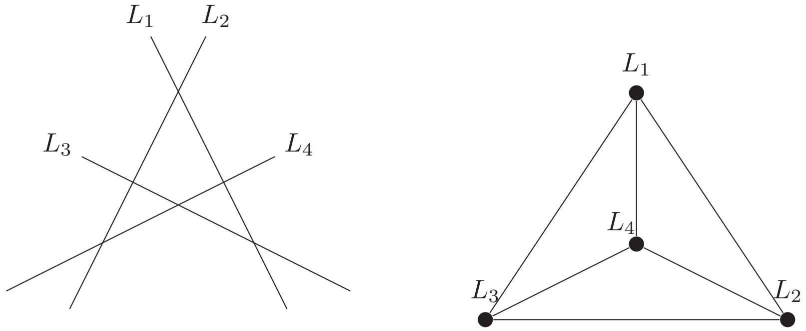

Figure III.1: Curve arrangement from Example III.3.1 and its boundary complex $\Delta_{\partial X}$.

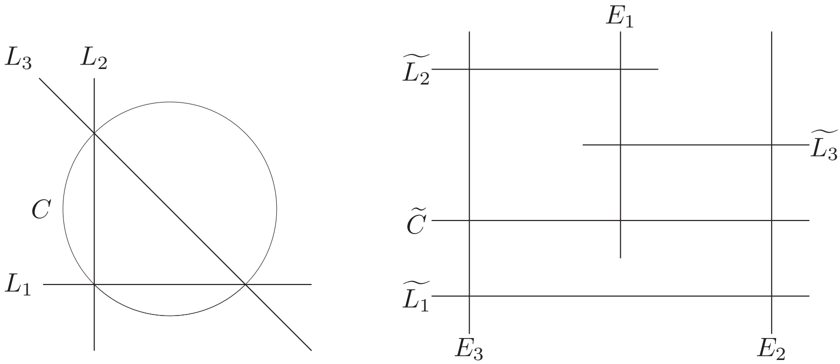

Figure III.2: Curve arrangement from Example III.3.2 and the compactifying divisor.

To compute the tropicalization, it suffices to compute the rays ρ_i corresponding to each of the irreducible components L_i of ∂X. The primitive integer vector v_i along ρ_i is the image of the basis element $e_j \in \bigoplus_{D_i \in \partial X} \mathbb{Z} = \mathbb{Z}^4$ under the quotient by $\operatorname{Pic}(\mathbb{P}^2_K)^\vee$, which is given by the map $(1,1,1,1)$. Taking $e_1 = -e_2 - e_3 - e_4$, we obtain a new basis for the quotient space $\mathbb{Z}^4 / \operatorname{Pic}(\mathbb{P}^2_K)^\vee$, and the rays of the fan $\operatorname{Trop}(X)$ are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,-1,-1)$.

Example III.3.2. Consider the arrangement of curves consisting of the coordinate axes $L_1, L_2, L_3 \subset \mathbb{P}^2_K$, as well as the conic C passing through the three points of intersection $L_i \cap L_j$, as shown in Figure III.2. The complement of the arrangement $X \coloneqq \mathbb{P}^2_K \smallsetminus (L_1 \cup L_2 \cup L_3 \cup C)$ is not compactified by a simple normal crossing divisor in $\mathbb{P}^2_K$, so we must blow up the three points $L_i \cap L_j$. This yields a configuration including three exceptional divisors E_1, E_2 and E_3, also displayed in Figure III.2.

Blowing up the plane in the three points $\mathbb{P}^2_K$ to compactify X into $\overline{X}$ yields the Picard group $\operatorname{Pic}(\overline{X}) \cong \langle H \rangle \oplus_{i=1}^3 \langle e_j \rangle$, where H is the pullback of the line class in $\mathrm{Pic}(\mathbb{P}^2_K)$. Moreover, the boundary ∂X consists of seven curves, and the intrinsic torus $\mathcal{O}(X)^*/K^*$ is of dimension three. The short exact sequence from (III.1) is then

$$0 \to \langle H \rangle \oplus_{i=1}^{3} \langle e_j \rangle \xrightarrow{\phi} \oplus_{i=1}^{3} \langle \widetilde{L_i} \rangle \oplus \langle \widetilde{C} \rangle \oplus_{i=1}^{3} \langle e_j \rangle \xrightarrow{\pi} \mathcal{O}(X)^*/K^* \to 0,$$

where the lattice map ϕ is given by the matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 2 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To compute the rays corresponding to each divisor D of the boundary ∂X, we consider $\mathcal{O}(X)^*/K^*$ as the cokernel of ϕ under the quotient map π. Somewhat abusing notation, each column yields a relation in $\mathcal{O}(X)^*/K^*$ as follows

$$\begin{aligned} \pi(E_1) &= & \pi(\widetilde{L_2}) + \pi(\widetilde{L_3}) + \pi(\widetilde{C}) \\ \pi(E_2) &= \pi(\widetilde{L_1}) & + \pi(\widetilde{L_3}) + \pi(\widetilde{C}) \\ \pi(E_3) &= \pi(\widetilde{L_1}) + \pi(\widetilde{L_2}) & + \pi(\widetilde{C}) \end{aligned}$$

as well as the equality

$$\pi(\widetilde{L_1}) + \pi(\widetilde{L_2}) + \pi(\widetilde{L_3}) + 2\pi(\widetilde{C}) = 0.$$

In particular, the rays of the fan corresponding to the divisors are given by the columns of the following matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -2 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix},$$

and embedding the dual complex $\Delta_{\partial X}$ as the cone over these rays, we observe that multiple of the rays are contained inside a two-dimensional face, and we are left with only the four rays

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

which gives a fan with support equal to the tropicalization of X.

III.4 Axioms for abstract arrangements of curves

In light of the description of the tropicalization of an arrangement of curves in Section III.3, we now propose an axiomatization generalizing arrangements of curves. We show that for a given arrangement, the axiomatization data extracted from the arrangement is sufficient to compute its tropicalization in its intrinsic torus.

III.4.1 Axioms

Let $\mathcal{A} \coloneqq \{1, \ldots, n\}$ be a finite set, called the *ground set* or *divisors*. Let r be an integer, either 1 or 2, called the *(reduced) rank*. There is a *degree* function $d\colon \mathcal{A} \to \mathbb{Z}_{>0}$, and for each element $i \in \mathcal{A}$, we will use the notation $d_i \coloneqq d(i)$. Let $\mathcal{P}$ be a multiset of subsets of $\mathcal{A}$, called the *points*. Each point $\mathbf{p} \in \mathcal{P}$ is equipped with an *intersection multiplicity function* $m_{\mathbf{p}}\colon \mathbf{p}^r \to \mathbb{Z}$, which is such that

- for $(i_1, \ldots, i_r) \in \mathbf{p}^r$ a vector, there are bounds $1 \leq m_{\mathbf{p}}(i_1, \ldots, i_r) \leq \max_{k \in \{i_1, \ldots, i_r\}}(d_k)$, and
- $m_{\mathbf{p}}$ is invariant under permutation of the coordinates.

The multiset of points $\mathcal{P}$ satisfies the *Bézout property* if for all vectors $(i_1, \ldots, i_r) \in \mathcal{A}^r$, there are exactly $d_{i_1} d_{i_2} \cdots d_{i_r}$ points of $\mathcal{P}$ containing $\{i_1, \ldots, i_r\}$, when counting with intersection multiplicity, i.e.

$$\sum_{\mathbf{p} \supseteq \{i_1, \ldots, i_r\}} m_{\mathbf{p}}(i_1, \ldots, i_r) = d_{i_1} d_{i_2} \cdots d_{i_r},$$

where the sum is taken over all points $\mathbf{p} \in \mathcal{P}$. Let $m \coloneqq \{m_{\mathbf{p}} \mid \mathbf{p} \in \mathcal{P}\}$ denote the set of multiplicity functions associated to $\mathcal{P}$.

Definition III.4.1. An *arroid* is a tuple $(\mathcal{A}, d, \mathcal{P}, m)$ satisfying the Bézout property.

The data contained in an arroid of rank two records the intersection properties of an arrangement of curves in the plane. More generally, any loop-free rank three matroid gives rise to a rank two arroid, as shown in the following example.

Example III.4.2. Let M be a loop-free matroid of rank three on a ground set $E \coloneqq \{1, \ldots, n\}$, with rank function $r\colon 2^E \to \mathbb{N}$. We construct an arroid $\mathbf{A}$ as follows. The ground set $\mathcal{A}$ of $\mathbf{A}$ is taken to be E, equipped with the degree function d taking constant value 1. The multiset of points $\mathcal{P}$ of this arroid is exactly the set of rank two flats of the matroid, where the multiplicity functions are also constant of value 1. For the Bézout property take a pair (i, j) of elements in the ground set, and note that since M is loop-free, $\{i\}$ is a flat. By the flat partition property, there is exactly one rank three flat containing j, thus there is exactly one rank three flat, i.e. point of the arroid, containing $\{i, j\}$ as required.

For more general examples, consider the following:

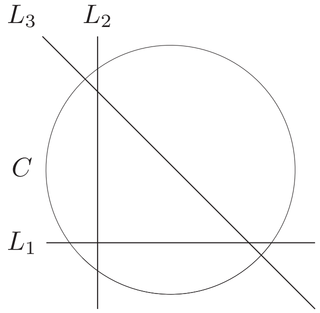

Figure III.3: Curve arrangement from Example III.4.4.

Example III.4.3. We return to Example III.3.2. Let $\mathcal{A} = \{1, 2, 3, 4\}$, where $i = 1, 2, 3$ corresponds to the line L_i, and 4 corresponds to the conic C. The degree function $d\colon \mathcal{A} \to \mathbb{Z}_{>0}$ is given by $d_i = 1$ for $i < 4$ and $d_4 = 2$. The multiset of points is $\mathcal{P} = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, corresponding to the set of intersection points of the lines and conic. The Bézout property of $\mathcal{P}$ follows directly from the Bézout theorem for $\mathbb{P}^2_K$.

Example III.4.4. Consider now the generic arrangement of three lines and a conic displayed in Figure III.3. As before, let $\mathcal{A} = \{1, 2, 3, 4\}$, where $i = 1, 2, 3$ corresponds to the line L_i, and 4 corresponds to the conic C, with the degree function $d\colon \mathcal{A} \to \mathbb{Z}_{>0}$ given by $d_i = 1$ for $i < 4$ and $d_4 = 2$. The multiset structure of the points is now more evident as

$$\mathcal{P} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{1, 4\}, \{2, 4\}, \{2, 4\}, \{3, 4\}, \{3, 4\}\},$$

which reflects that each pair of lines meets in a single point, and the conic meets each line in two points.

In general, given an arrangement of curves $\mathcal{B} = \{C_1, \dots, C_n\}$ on $\mathbb{P}^2_K$, we can create an arroid of rank two as follows. Let $\mathcal{A} \coloneqq \{1, \dots, n\}$ and the function d takes i to the degree of the curve d_i. For any subset $\mathbf{p} \subseteq \mathcal{B}$ such that $C_I = \cap_{i \in I} C_i \neq \emptyset$, let $\mathbf{p}_{I_1}, \dots, \mathbf{p}_{I_k}$ be the components of C_I. For each $\mathbf{p}_{I_j}$, we include $\mathbf{p}$ as an element in $\mathcal{P}$ with multiplicity function $m_{\mathbf{p}}$ given by the intersection multiplicities of the curves at the component $\mathbf{p}_{I_j}$. Thus the point $\mathbf{p}$ may appear multiple times with potentially different multiplicity functions. We denote the arroid of an arrangement $\mathcal{B}$ by $\mathbf{A}_{\mathcal{B}}$.

An arroid of rank one should be viewed as formalizing the process of restricting an arrangement of curves to considering the points of the arrangement on one of the given curves, while recording exactly which curves pass through a given point. In fact we may formalize this in the notion of an *arroid contraction*, contracting a rank $r = 2$ arroid to a rank $r = 1$ arroid. Let $\mathbf{A} = (\mathcal{A}, d, \mathcal{P}, m)$ be a rank 2 arroid, and $i \in \mathcal{A}$ a divisor. The *contraction* $\mathbf{A}/i$ of $\mathbf{A}$ is the arroid $\mathbf{A}/i \coloneqq (\mathcal{A} \smallsetminus \{i\}, d/i, \mathcal{P}/i, m^{\mathbf{A}/i})$, where $d/i(j) \coloneqq d_i \cdot d(j)$, and $\mathcal{P}/i$ is the multiset

$\{\mathbf{p} \smallsetminus \{i\} \mid i \in \mathbf{p}\}_{\mathbf{p} \in \mathcal{P}}$. The multiplicity functions, now of rank $r = 1$, are the function

$$m^{\mathbf{A}/i}_{\mathbf{p} \smallsetminus \{i\}} \colon \mathbf{p} \smallsetminus \{i\} \to \mathbb{Z}$$

given by mapping an element $j \in \mathbf{p} \smallsetminus \{i\}$ to $m_{\mathbf{p}}(i, j)$, i.e. using the multiplicity function of the original points $\mathbf{p}$. The Bézout condition for $\mathbf{A}/i$ follows from that for $\mathbf{A}$.

Furthermore, in the context of arrangements of curves, one may always consider the arrangement obtained by removing one of the curves. In the arroid context, we generalize this notion in the form of an *arroid deletion*. The *deletion* $\mathbf{A} \smallsetminus i$ of and arroid $\mathbf{A}$ is the arroid $\mathbf{A} \smallsetminus i := (\mathcal{A} \smallsetminus \{i\}, d, \mathcal{P} \smallsetminus i, m^{\mathbf{A} \smallsetminus i})$, where d takes the same values as for d of $\mathbf{A}$, and $\mathcal{P} \smallsetminus i$ is the multiset consisting of the points $\mathbf{p}$ of $\mathcal{P}$ not containing i, as well as points $\mathbf{p} \smallsetminus \{i\}$ where $\mathbf{p} \neq \{i, j\}$ for some $j \in \mathcal{A}$. Since each point $\mathbf{q}$ of $\mathbf{A} \smallsetminus i$ corresponds to a specific point $\mathbf{p}$ of $\mathbf{A}$, we define the multiplicity function

$$m^{\mathbf{A} \smallsetminus i}_{\mathbf{q}} \colon \mathbf{q}^2 \to \mathbb{Z}$$

as given by $m^{\mathbf{A} \smallsetminus i}_{\mathbf{q}}(j, k) = m^{\mathbf{A}}_{\mathbf{p}}(j, k)$ for all $j, k \in \mathbf{q}$.

Finally, we describe a certain type of arroid, which is inspired by the case of an arrangement of curves where all the curves intersect pairwise transversely.

Definition III.4.5. A *transversal arroid* is an arroid such that for all $\mathbf{p} \in \mathcal{P}$, the intersection multiplicity $m_{\mathbf{p}}$ is constant taking value 1.

Note that the arroid deletion of a transversal arroid is itself transversal.

III.4.2 The fan of a transversal arroid

Let $\mathbf{A} = (\mathcal{A}, d, \mathcal{P}, m)$ be a transversal arroid. We will now construct a weighted rational polyhedral fan $\Sigma_{\mathbf{A}} \subset \mathbb{R}^{|\mathcal{A}|}/\langle (d_1, \dots, d_n) \rangle$ associated to a transversal arroid. This is inspired by the construction of the Bergman fan as the cone over the order complex of matroids as done in [AK06]. Let $[w]$ denote the class in $\mathbb{R}^{|\mathcal{A}|}/\langle (d_1, \dots, d_n) \rangle$ of the vector $w \in \mathbb{R}^{|\mathcal{A}|}$, and $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^{|\mathcal{A}|}$. Denote by $\mathcal{P}^u$ the underlying set of the multiset $\mathcal{P}$, i.e. discarding the number of occurrences of each point $\mathbf{p} \in \mathcal{P}$ and recording it only once in $\mathcal{P}^u$.

First consider $\mathbf{A}$ an arroid of rank one, and let $\Sigma_{\mathbf{A}} \subset \mathbb{R}^{|\mathcal{A}|}/\langle (d_1, \dots, d_n) \rangle$ be the following weighted one-dimensional rational polyhedral fan. For each point $\mathbf{p} \in \mathcal{P}^u$, let $v_{\mathbf{p}} := \sum_{j \in \mathbf{p}} [e_j]$ be a ray of $\Sigma_{\mathbf{A}}$. The weight function $\omega \colon \Sigma_1 \to \mathbb{Z}$ is defined by taking $v_{\mathbf{p}}$ to $w(\mathbf{p})$, where $w(\mathbf{p})$ is the number of times $\mathbf{p}$ is repeated in the multiset $\mathcal{P}$. This fan is *tropical*, in the sense that it satisfies the balancing condition in codimension one [BIMS15, Definition 5.7], as we have

$$\sum_{\mathbf{p} \in \mathcal{P}^u} \omega(\mathbf{p}) v_{\mathbf{p}} = \sum_{\mathbf{p} \in \mathcal{P}} \sum_{j \in \mathbf{p}} [e_j] = [\sum_{j \in \mathcal{A}} \sum_{\substack{\mathbf{p} \in \mathcal{P} \\ j \in \mathbf{p}}} e_j] = [\sum_{j \in \mathcal{A}} d_j e_j] = 0,$$

where we used the Bézout property and the transversality to conclude that $\sum_{j \in \mathbf{p}} 1 = d_j$.

Now we turn to arroids $\mathbf{A}$ of rank two. For each divisor $i \in \mathcal{A}$, the cone $\rho_i := \operatorname{cone}([e_j])$ is a ray of $\Sigma_{\mathbf{A}}$. For each point $\mathbf{p} \in \mathcal{P}^u$, the cone $\rho_{\mathbf{p}} := \operatorname{cone}(v_{\mathbf{p}})$, with $v_{\mathbf{p}} := \sum_{i \in \mathbf{p}} [e_j]$, is also a ray of $\Sigma_{\mathbf{A}}$. The two-dimensional cones of $\Sigma_{\mathbf{A}}$ are the following. For each pair $i \in \mathcal{A}$ and $\mathbf{p} \in \mathcal{P}$ with $i \in \mathbf{p}$, there is a two-dimensional cone $\sigma_{i,\mathbf{p}} := \operatorname{cone}(\rho_i, \rho_{\mathbf{p}})$ in $\Sigma_{\mathbf{A}}$. The weight function $\omega \colon \Sigma_2 \to \mathbb{Z}$ is defined by taking $\sigma_{i,\mathbf{p}}$ to $w(\mathbf{p})$, where $w(\mathbf{p})$ is the number of times $\mathbf{p}$ is repeated in the multiset $\mathcal{P}$.

This fan is also tropical. Now we must verify balancing at each ray. There are two types of rays to verify: the $\rho_{\mathbf{p}}$ and the ρ_i. Picking a ray $\rho_{\mathbf{p}}$, we must verify that the sum $\sum_{\sigma_{i,\mathbf{p}} \succ \rho_{\mathbf{p}}} \omega(\sigma_{i,\mathbf{p}}) v_{\sigma_{i,\mathbf{p}}/\rho_{\mathbf{p}}}$ is a vector contained in $\rho_{\mathbf{p}}$, where $v_{\sigma_{i,\mathbf{p}}/\rho_{\mathbf{p}}}$ together with the primitive vector $v_{\mathbf{p}}$ of $\rho_{\mathbf{p}}$ generates $\sigma_{i,\mathbf{p}}$. One such $v_{\sigma_{i,\mathbf{p}}/\rho_{\mathbf{p}}}$ is precisely $[e_j]$, the primitive vector of ρ_i, and so we have

$$\sum_{\sigma_{i,\mathbf{p}} \succ \rho_{\mathbf{p}}} \omega(\sigma_{i,\mathbf{p}}) v_{\sigma_{i,\mathbf{p}}/\rho_{\mathbf{p}}} = \sum_{i \in \mathbf{p}} w(\mathbf{p})[e_j] = w(\mathbf{p})[\sum_{i \in \mathbf{p}} e_j] = w(\mathbf{p})[v_\rho],$$

which is contained in $\rho_{\mathbf{p}}$. The balancing condition for the rays ρ_i follows by using the Bézout condition in the following manner. Picking a ray ρ_i, we must verify that the sum $\sum_{\sigma_{i,\mathbf{p}} \succ \rho_i} \omega(\sigma_{i,\mathbf{p}}) v_{\sigma_{i,\mathbf{p}}/\rho_i}$ is a vector contained in ρ_i, where $v_{\sigma_{i,\mathbf{p}}/\rho_i}$ together with the primitive vector $[e_j]$ of ρ_i generates $\sigma_{i,\mathbf{p}}$. We pick $v_{\sigma_{i,\mathbf{p}}/\rho_i} = [v_{\mathbf{p}}]$, then

$$\sum_{\sigma_{i,\mathbf{p}} \succ \rho_i} \omega(\sigma_{i,\mathbf{p}}) v_{\sigma_{i,\mathbf{p}}/\rho_i} = \sum_{\substack{\mathbf{p} \ni i \\ \mathbf{p} \in \mathcal{P}^u}} w(\mathbf{p})[v_{\mathbf{p}}] = \sum_{\substack{\mathbf{p} \ni i \\ \mathbf{p} \in \mathcal{P}}} [\sum_{j \in \mathbf{p}} e_j] = \sum_{\substack{j \in \mathcal{A} \\ j \neq i}} \sum_{\substack{\mathbf{p} \supset \{i,j\} \\ \mathbf{p} \in \mathcal{P}}} [e_j] + \sum_{\substack{\mathbf{p} \ni i \\ \mathbf{p} \in \mathcal{P}}} [e_j].$$

Now by the Bézout condition, each vector $e_j \neq e_i$ appears exactly $d_i d_j$ times, so the first term is equivalent to a multiple of $[e_i]$ in the quotient $\mathbb{R}^{|\mathcal{A}|}/\langle (d_1, \ldots, d_n) \rangle$. This implies in particular that the whole sum is equal to some multiple of $[e_i]$, and hence contained in ρ_i.

By the above constructions, we have associated a fan $\Sigma_{\mathbf{A}}$ for each transversal arroid $\mathbf{A}$, which we call the *arroid fan* of $\mathbf{A}$. Moreover, taking all the balancing properties into account, we have proven the following.

Theorem III.4.6. *For each transversal arroid $\mathbf{A}$, the arroid fan $\Sigma_{\mathbf{A}}$ is a balanced tropical variety.*

Remark III.4.7. It follows from the above construction that the fan of a transversal arroid of rank one or two is necessarily *connected through codimension one* in the sense of [MS15, Definition 3.3.4].

Furthermore, we describe the reduced star at the rays of the fan $\Sigma_{\mathbf{A}}$ an arroid $\mathbf{A}$ of rank two in terms of arroid fans.

Proposition III.4.8. *Let $\Sigma_{\mathbf{A}}$ be an arroid fan, and $i \in \mathcal{A}$ correspond to the ray ρ_i of $\Sigma_{\mathbf{A}}$. Then the reduced star ${\Sigma_{\mathbf{A}}}^{\rho_i}$ is equal to the arroid fan $\Sigma_{\mathbf{A}/i}$ of the arroid contraction $\mathbf{A}/i$.*

Proof. This follows by considering both definitions. The fan $\Sigma_{\mathbf{A}/i}$ is contained in the associated space $\mathbb{R}^{|\mathcal{A} \smallsetminus \{i\}|}/\langle (d_1, \ldots, \hat{d}_i, \ldots, d_n) \rangle$, which can also be viewed as the space $(\mathbb{R}^{|\mathcal{A}|}/\langle(d_1,\ldots,d_n)\rangle)/\langle e_i\rangle$ that contains ${\Sigma_{\mathbf{A}}}^{\rho_i}$. Moreover, the cones of both these fans correspond exactly to the projection of the cones of Σ containing ρ_i, i.e. the points $\mathbf{p}\in\mathcal{P}$ containing i. ■

Next, we show that the fan of a transversal arroid may be constructed inductively using tropical modifications along any of its arroid deletions. This is inspired by the construction of the Bergman fan of a matroid through tropical modifications described in [Sha11, Proposition 1.1.34], as well as the *shellability* property in [AP21].

Proposition III.4.9. *Let $\mathbf{A}=(\mathcal{A},d,\mathcal{P},m)$ be a transversal arroid, and $\mathbf{A}\setminus i$ one of its arroid deletions. Then $\Sigma_{\mathbf{A}}=\mathrm{TM}(\Sigma_{\mathbf{A}\setminus i},\Sigma_{\mathbf{A}/i})$ is a tropical modification of the fan of the arroid deletion $\mathbf{A}\setminus i$ along the fan of the arroid contraction $\mathbf{A}/i$.*

Proof. We will show that the preimage of a point in the fan $\Sigma_{\mathbf{A}}$ under the coordinate projection $\pi\colon\mathbb{R}^{|\mathcal{A}|}/\langle(d_1,\ldots,d_n)\rangle\to\mathbb{R}^{|\mathcal{A}\setminus i|}/\langle(d_1,\ldots,\hat{d_i},\ldots,d_n)\rangle$ is either a half-line in the e_i direction, or a point. Then one may obtain a piecewise integer affine function g defining the tropical modification by taking $g(p)=\pi^{-1}(p)$ for the points $p\in\Sigma_{\mathbf{A}\setminus i}$ whose preimage $\pi^{-1}(p)$ are single points.

To show that $\pi^{-1}(p)$ is either a point or half-line in $\Sigma_{\mathbf{A}}$, we analyze the cones of the latter. The cones of $\Sigma_{\mathbf{A}}$ have one of three forms:

- $\mathrm{cone}(e_j,e_j{+}e_i)$ for some $j\in\mathcal{A}$, corresponding to a point $\mathbf{p}$ which disappears in $\mathbf{A}\setminus i$,
- $\mathrm{cone}(e_j,e_{\mathbf{a}}+e_i)$, corresponding to a point $\mathbf{p}=\mathbf{a}\cup i\in\mathcal{P}$ from which i is removed in $\mathbf{A}\setminus i$, or
- $\mathrm{cone}(e_j,e_{\mathbf{p}})$ for some $\mathbf{p}\in\mathcal{P}$ not containing i.

Any point p of the fan $\Sigma_{\mathbf{A}\setminus i}$ is contained inside $\mathrm{cone}(e_k,e_{\mathbf{p}})$ for some k and $\mathbf{p}\in\mathcal{P}\setminus i$ with $k\in\mathbf{p}$. We distinguish multiple cases:

- If $p=\mathbf{0}$ is the origin, then $\pi^{-1}(p)=\mathrm{cone}(e_i)$ is a half-ray of the fan $\Sigma_{\mathbf{A}}$.
- If $p=\alpha e_k+\beta e_{\mathbf{p}}$ with $\alpha,\beta>0$, and $\mathbf{p}\cup\{i\}$ is a point of $\mathbf{A}$, the preimage $\pi^{-1}(p)$ is $p'=\alpha e_j+\beta(e_{\mathbf{p}}+e_i)$ in $\Sigma_{\mathbf{A}}$.
- If $p=\alpha e_k$ then either $\{i,k\}\in\mathcal{P}$, and then $\pi^{-1}(p)=\{ae_j+b(e_j+e_i)\mid a+b=\alpha;\ a,b\geq 0\}$ is a half-ray or $\{i,k\}\notin\mathcal{P}$ and then $\pi^{-1}(p)=\alpha e_k$.
- If $p=\beta e_{\mathbf{p}}$ then either $\mathbf{p}\cup\{i\}$ is a point of $\mathbf{A}$, and the preimage is a half.line, or it is not so that $\pi^{-1}(p)=\beta e_{\mathbf{p}}$.

For any of p, the fiber is either a half-line or a point, and thus $\Sigma_{\mathbf{A}}$ is a tropical modification of $\Sigma_{\mathbf{A}\setminus i}$.

Moreover, it follows from the construction that the divisor along which the modification is performed is the reduced star ${\Sigma_{\mathbf{A}}}^{\rho_i}$, which is $\Sigma_{\mathbf{A}/i}$ by Proposition III.4.8, so that we have $\Sigma_{\mathbf{A}}=\mathrm{TM}(\Sigma_{\mathbf{A}\setminus i},\Sigma_{\mathbf{A}/i})$. Moreover, the weights of $\Sigma_{\mathbf{A}}$ are then given in terms of the weights of the rays of $\Sigma_{\mathbf{A}/i}$, which is compatible with the construction of both fans as above. ■

III.4.3 Tropicalization

We now show that the data gathered in the transversal arroid of an arrangement of curves is sufficient to compute its tropicalization. It is in fact sufficient to note that, all the data used in the description of the tropicalization given in Section III.3 is recorded in the arroid. Our aim is instead to show that, for an arrangement where all intersections are transverse, the fan described in Section III.4.2 is a fan structure on the tropicalization of the arrangement.

First, to insure that the arrangement of curves $\mathcal{B}$ in the plane $\mathbb{P}^2_K$ is in fact very affine, we suppose that it contains at least three lines intersecting generically, so that they can be mapped to the coordinate axes by a projective transformation. We will say that such an arrangement is *very affine*. If all the curves intersect pairwise transversely, we will say that the arrangement $\mathcal{B}$ is *transverse*. In this case, the associated arroid is transversal because the multiplicity functions at each point of the arroid $\mathbf{A}_{\mathcal{B}}$ is given by the intersection multiplicities of the curves of $\mathcal{B}$ at the corresponding point. Since all the curves intersect pairwise transversely, the multiplicity function $m_{\mathbf{p}}$ is constant of value 1, hence the arroid is transversal.

Theorem III.4.10. *Let $\mathcal{B}$ be a transverse very affine arrangement of curves in the plane $\mathbb{P}^2_K$. Then the tropicalization* $\operatorname{trop}(X_{\mathcal{B}})$ *of the complement is supported on the fan of the associated transversal arroid $\mathbf{A}_{\mathcal{B}}$.*

Proof. We show that the arroid fan is supported on the tropicalization of the complement of the arrangement. Blowing up $\mathbb{P}^2_K$ in any point where more than 3 of the curves in $\mathcal{B}$ meet, we obtain a simple normal crossing divisor D which compactifies the complement $X_{\mathcal{B}} := \mathbb{P}^2_K \setminus \cup_{C\in\mathcal{B}} C$. Therefore, using the map to the intrinsic torus, Theorem III.2.2 gives us that the tropicalization is equal to the fan whose rays are $\operatorname{cone}([\operatorname{val}_D])$ in $N_{\mathbb{R}}$ for each irreducible component D of the simple normal crossing divisor $\mathbf{D}$, and whose two-dimensional cones are $\operatorname{cone}([\operatorname{val}_D], [\operatorname{val}_{D'}])$ for boundary divisors D and D' such that $D \cap D' \neq \emptyset$.

There are two types of irreducible components of $\mathbf{D}$: the strict transforms of the curves C in B, whose divisorial valuations are such that the associated vectors $[\operatorname{val}_C]$ is a standard basis vector of $N_{\mathbb{R}}$, and the exceptional divisors E of the blown-up loci $\mathbf{p}$, for which the valuation val_E computes the order of vanishing at $\mathbf{p}$. Therefore, the vector $[\operatorname{val}_E]$ is the sum $\sum_{i_k} [\operatorname{val}_{D_{i_k}}]$ for the D_{i_k} which intersect at $\mathbf{p}$. In particular, it follows from this description of the rays that the tropicalization is exactly equal to the fan of the arroid $\mathbf{A}_{\mathcal{B}}$. ■

It follows from the definitions that, given the arroid $\mathbf{A}$ of an arrangement of curves $\mathcal{B}$, and considering the arrangement obtained by removing one of the curves, say numbered i, from the arrangement $\mathcal{B}$, the corresponding arroid will be the deletion $\mathbf{A} \setminus i$.

III.5 Tropical homology manifold arroid fans

In this section, we study when a transversal arroid fan is a tropical homology manifold. We will first show that all tropical Borel–Moore homology of a transversal arroid fan is concentrated in top degree, and hence torsion-free. Then we will use an Euler characteristic argument to show that being a tropical homology manifold reduces to a balancing condition. This property is a main component in understanding when the complement of an arrangement of curves is cohomologically tropical, see Theorem III.6.2.

III.5.1 Tropical cohomology of transversal arroid fans

For a transversal arroid, the main structural result is that the tropical Borel–Moore homology of the associated fan is concentrated in top-degree. More precisely, we have the following lemma.

Lemma III.5.1. *Let $\mathbf{A}$ be a transversal arroid. Then $H^{BM}_{p,q}(\Sigma_{\mathbf{A}}) = 0$ for $q \neq 2$, for all p.*

Proof. By Proposition III.4.9, each arroid fan $\Sigma_{\mathbf{A}}$ is the tropical modification of any of its deletions, by adding a divisor to the ground set. Hence it suffices to show that the property holds for the modification of an arroid fan. Let $\Sigma_{\mathbf{A}}$ be an arroid fan, and $\mathrm{TM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))$ a modification of $\Sigma_{\mathbf{A}}$. As pointed out in the proof of [JRS18, Proposition 5.5], there is a short exact sequence of complexes

$$\begin{aligned} 0 \to C^{BM}_{p,\bullet}(\mathrm{tropdiv}(\phi)) \to C^{BM}_{p,\bullet}&(\mathrm{CTM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))) \\ &\to C^{BM}_{p,\bullet}(\mathrm{TM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))) \to 0, \end{aligned}$$

and moreover, by Lemma III.5.3, the long exact sequence in homology takes the form

$$\begin{aligned} \cdots \to H^{BM}_{p,q}(\mathrm{tropdiv}(\phi)) \to H^{BM}_{p,q}(\Sigma_{\mathbf{A}}) \to H^{BM}_{p,q}&(\mathrm{TM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))) \\ &\to H^{BM}_{p,q-1}(\mathrm{tropdiv}(\phi)) \to \cdots . \end{aligned}$$

By induction we have $H^{BM}_{p,q}(\Sigma_{\mathbf{A}}) = 0$ for $q \neq 2$, and similarly $H^{BM}_{p,q}(\mathrm{tropdiv}(\phi)) = 0$ for $q \neq 1$, since $\mathrm{tropdiv}(\phi)$ is one-dimensional. Therefore the result follows by exactness. ∎

Lemma III.5.2 ([Sha11, Lemma 2.2.7])**.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a tropical fan, and $\mathrm{TM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))$ its tropical modification along the tropical divisor $\mathrm{tropdiv}(\phi)$. Then for each $p = 1, \dots, \dim \Sigma$, we have short exact sequences*

$$0 \longrightarrow H_{p-1,0}(\mathrm{tropdiv}(\phi)) \xrightarrow{\gamma} H_{p,0}(\mathrm{TM}(\Sigma, \mathrm{tropdiv}(\phi))) \xrightarrow{\delta} H_{p,0}(\Sigma) \longrightarrow 0,$$

$$0 \longrightarrow H^{p,0}(\Sigma) \xrightarrow{\delta^\vee} H^{p,0}(\mathrm{TM}(\Sigma, \mathrm{tropdiv}(\phi))) \xrightarrow{\gamma^\vee} H^{p-1,0}(\mathrm{tropdiv}(\phi)) \longrightarrow 0,$$

dual to each other, where γ is the map $w \mapsto w \wedge e_{n+1}$ and δ is $v \mapsto \pi(v)$, for $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n components.

Proof. The [Sha11, Lemma 2.2.7] is stated for tropical modifications of matroid fans, however the proof is applicable to this more general context. ■

Lemma III.5.3. *Let $\Sigma \subseteq \mathbb{R}^n$ be a tropical fan, and* $\mathrm{CTM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi))$ *its closed tropical modification along the tropical divisor* $\mathrm{tropdiv}(\phi)$. *Then* $H^{BM}_{p,q}(\Sigma_{\mathbf{A}}) \cong H^{BM}_{p,q}(\mathrm{CTM}(\Sigma_{\mathbf{A}}, \mathrm{tropdiv}(\phi)))$ *for all p and q.*

Proof. The [JRS18, Lemma 5.7] holds for arbitrary tropical modifications by Lemma III.5.2, and thus the proof given for [JRS18, Proposition 5.6] generalizes to the present context. ■

Note that the above strategy is applicable also in the case of $\mathbb{Z}$-coefficients for tropical homology, which shows that there is no torsion in the tropical Borel–Moore homology of the fan of a transversal arroid. Moreover, it follows that the fan of a transversal arroid is a tropical homology manifold if and only if it is uniquely balanced along each ray.

Theorem III.5.4. *Let $\Sigma_{\mathbf{A}}$ be the arroid fan of a transversal arroid. Then $\Sigma_{\mathbf{A}}$ is a tropical homology manifold if and only if it is uniquely balanced along each of its rays.*

Proof. By Lemma III.5.1, the tropical Borel–Moore groups are such that $H^{BM}_{p,q}(\Sigma_{\mathbf{A}}) = 0$ for $q \neq 2$, for all p. Moreover, for each cone γ of Σ, the reduced star fan Σ^γ has $H^{BM}_{p,q}(\Sigma^\gamma) = 0$ for $q \neq 1$, for all p, since Σ^γ is one-dimensional. Therefore, the result follows by [Aks23, Theorem 5.10]. ■

Finally, we conclude with an argument showing that for arroids, the deletion operation preserves the tropical homology manifold property of the fan.

Lemma III.5.5. *Let $\mathbf{A}$ be a transversal arroid such that the arroid fan $\Sigma_{\mathbf{A}}$ is a tropical homology manifold. Then for any arroid deletion $\mathbf{A} \setminus i$, the corresponding arroid fan $\Sigma_{\mathbf{A}\setminus i}$ is a tropical homology manifold.*

Proof. Since $\Sigma_{\mathbf{A}}$ is a tropical homology manifold, the reduced star along the ray corresponding to i, which by Proposition III.4.8 is the arroid fan $\Sigma_{\mathbf{A}/i}$ of the contraction $\mathbf{A}/i$, satisfies Tropical Poincaré Duality. By Proposition III.4.9, we have that $\Sigma_{\mathbf{A}} = \mathrm{TM}(\Sigma_{\mathbf{A}\setminus i}, \Sigma_{\mathbf{A}/i})$, so that for each $p = 0, 1, 2$, [JRS18, Diagram 5.6] takes the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{p,0}(\Sigma_{\mathbf{A}\setminus i}) & \longrightarrow & H^{p,0}(\Sigma_{\mathbf{A}}) & \longrightarrow & H^{p,0}(\Sigma_{\mathbf{A}/i}) & \longrightarrow & 0 \\
 & & \big\downarrow{\frown[\Sigma_{\mathbf{A}\setminus i}]} & & \big\downarrow{\frown[\Sigma_{\mathbf{A}}]} & & \big\downarrow{\frown[\Sigma_{\mathbf{A}/i}]} & & \\
0 & \longrightarrow & H^{BM}_{2-p,2}(\Sigma_{\mathbf{A}\setminus i}) & \longrightarrow & H^{BM}_{2-p,2}(\Sigma_{\mathbf{A}}) & \longrightarrow & H^{BM}_{2-p,2}(\Sigma_{\mathbf{A}/i}) & \longrightarrow & 0.
\end{array}$$

By assumption both the middle and rightmost vertical arrows are isomorphisms, and therefore so is the leftmost vertical arrow. Therefore $\Sigma_{\mathbf{A}\setminus i}$ satisfies tropical Poincaré duality, and so it remains to show that each reduced star ${\Sigma_{\mathbf{A}\setminus i}}^\gamma$ of a cone $\gamma \in \Sigma_{\mathbf{A}\setminus i}$ satisfies tropical Poincaré duality. In light of Lemma III.5.1, the result follows by applying [Aks23, Proposition 5.7]. ■

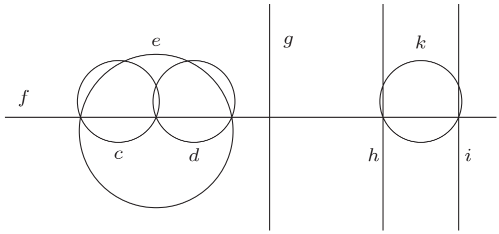

Figure III.4: A line with three clusters

III.5.2 Unique balancing of arroid rays

We now begin a preliminary investigation into when the rays of an arroid fan are uniquely balanced. In light of Theorem III.5.4, establishing such conditions would be equivalent to describing which arroid fans are tropical homology manifolds.

Note that the rays $\rho_{\mathbf{p}}$ corresponding to a point are always uniquely balanced. This follows since the equality

$$\sum_{i \in \mathbf{p}} \alpha_i [e_i] = \alpha[\nu_\rho]$$

holds if and only if $\alpha = \alpha_i$ for all i, where $\nu_\rho = \sum_{j \in \mathbf{p}} e_j$. Therefore the only balancing along such an edge is given by the weight $w(\mathbf{p})$ for each facet containing $\rho_{\mathbf{p}}$, as in Section III.4.2.

By the above, the only rays of an arroid fan for which unique balancing may fail, are the rays ρ_i corresponding to divisors $i \in \mathcal{A}$. For such a ray, a balancing corresponds to a set of weights $\alpha_{\mathbf{p}}$ for each $\mathbf{p}$ containing i, such that the equality

$$\sum_{\mathbf{p} \ni i} \alpha_{\mathbf{p}} [\nu_\rho] = \alpha[e_i] \tag{III.2}$$

holds, where the sum is taken over the set of unique points $\mathcal{P}^u$. Using the balancing already exhibited in Section III.4.2, we may increment both sides such that $\alpha_{\mathbf{p}} \geq 0$ for all $\mathbf{p}$.

We will now describe conditions on an arrangement of lines and conics which guarantee the unique balancing property along the rays corresponding to the lines. We begin with a certain connectedness notion for arrangements.

Definition III.5.6. Let $\mathcal{B}$ be a transverse very affine arrangement of curves in $\mathbb{P}^2_K$, and C a curve of the arrangement. A *cluster* of curves $\mathcal{C} \subset \mathcal{B}$ on C is a set of curves that for any two curves $C_a, C_b \in \mathcal{C}$, there is a sequence $C_a = C_1, C_2, \ldots, C_m = C_b$ such that $C_i \cap C_{i+1}$ contains a point of C.

Example III.5.7. Figure III.4 displays an arrangement where, for the line f, the sets of clusters are $\{e, c, d\}, \{k, h, i\}$ and the singleton $\{g\}$. The set $\{g, d\}$ is not a cluster for f.

A cluster may be a subset of another cluster, giving an ordering. For a chosen curve C, we may divide the points of the arrangement on C into the maximal clusters in which they are intersection points. For a line L in an arrangement of lines and conics, the following condition on the clusters of L implies unique balancing of the corresponding ray of the arroid fan.

Lemma III.5.8. *Let $\mathcal{B}$ be a transverse very affine arrangement of lines and conics in $\mathbb{P}^2_K$, and let $\mathbf{A}$ be the arroid of the arrangement. Let $i \in \mathcal{A}$ be a divisor corresponding to a line L. Suppose that each cluster of L contains a line. Then the arroid fan $\Sigma_{\mathbf{A}}$ is uniquely balanced along the ray ρ_i.*

Proof. We will show that for any cluster, the coefficients $\alpha_{\mathbf{p}}$ are all equal, for any point $\mathbf{p}$ on L of the cluster. For each cluster, let $\mathbf{p}_\ell$ be the point on L containing the line, and $\alpha_{\mathbf{p}_\ell}$ the corresponding coefficient in equation (III.2). Now for any of the conics C passing through this point, the Bézout condition, along with balancing implies that the other intersection point $\mathbf{p}'$ of C with L must have coefficient $\alpha_{\mathbf{p}'} = \alpha_{\mathbf{p}_\ell}$. By iterating through all pairs in the cluster, all coefficients of the points of the cluster are equal. Moreover, as this is true for all clusters of L, the balancing condition forces the coefficients of different clusters to be equal, making the balancing unique. ■

The unique balancing of rays associated with conics is seemingly more subtle. We first introduce specific types of clusters, from which we derive coarse criterion for unique balancing to occur.

Definition III.5.9. Let $\mathcal{B}$ be a transverse very affine arrangement of lines and conics in $\mathbb{P}^2_K$, and C a conic of the arrangement. A conic C' of a cluster $\mathcal{C}$ is a *source* if all four points of $C' \cap C$ are joined by at least five lines of the arrangement, and it called a *reservoir* if it is joined by at least four. For C'' another conic of the cluster, an *aqueduct* on C between C' and C'' is a line L of the arrangement joining a pair of intersection points in $C' \cap C$ and $C'' \cap C$. A cluster containing at least one source conic, with all other conics being reservoir conics, with a chain of aqueducts linking back to the source, is called a *balancing supply system.*

The sense in the above naming is that, for a source, all weights $\alpha_{\mathbf{p}}$ of the associated intersection points must be equal. If an aqueduct connects a source to a reservoir, then the weights of the intersection points at the reservoir must be equal to that of the source, and this equality is then spread further to all other reservoirs linked by aqueducts; i.e. if a cluster is a balancing supply system, all the weights of any balancing are equal.

Example III.5.10. In Figure III.5, we consider the conic C, which has a unique maximal cluster, comprised of all the lines together with the two additional conics C' and C''. Here C' is a source, C'' is a reservoir, and the line L is an aqueduct. Therefore the unique maximal cluster is a balancing supply system.

Lemma III.5.11. *Let $\mathcal{B}$ be a transverse very affine arrangement of lines and conics in $\mathbb{P}^2_K$, and let $\mathbf{A}$ be the arroid of the arrangement. Let $i \in \mathcal{A}$ be a divisor corresponding to a conic C. If all maximal cluster of C contain conics and are*

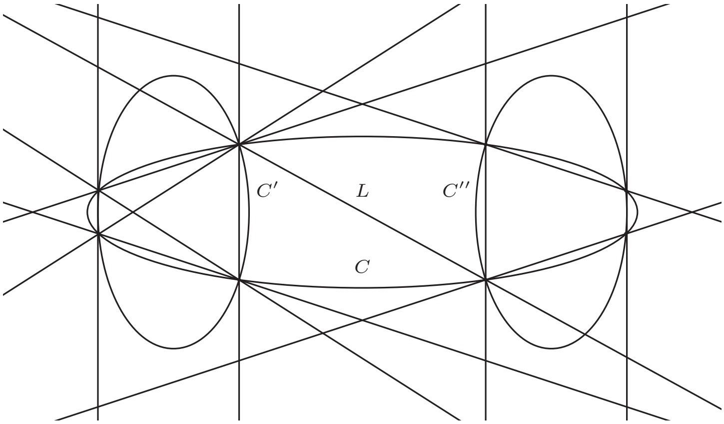

Figure III.5: A conic with balancing supply cluster

balancing supply system, then the arroid fan $\Sigma_{\mathbf{A}}$ *is uniquely balanced along the ray* ρ_i.

Proof. We follow the strategy of weight propagation suggested above. Let $\mathcal{C}$ be a maximal cluster, and C' a source. Since the arrangement is transverse, there are four intersection points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{p}_1$ in $C \cap C'$, and given that there are 5 lines passing through these points, they yield four distinct terms $\alpha_{\mathbf{p}}[\nu_{\mathbf{p}}]$ in the left hand sum (III.2). The balancing condition implies that the $\alpha_{\mathbf{p}}$ must all be equal, as otherwise the vectors $[e_j]$ corresponding to the lines do not appear an equal number of times. Next, let C'' be a reservoir, and $\mathbf{p} \in C'' \cap C$ a point where there is an aqueduct L between C' and C''. Then if the weights $\alpha_{\mathbf{p}_i} = \alpha$ are fixed, since the coordinate $[e_L]$ must appear 2α times, we have that $\alpha_{\mathbf{p}} = \alpha$ as well. Then for the same reason, lines passing through two intersection points of $C'' \cap C$ have components appearing 2α times, thus all points of $C'' \cap C$ must have the same weight. This reasoning will then also apply to any other reservoir connected to C'' by an aqueduct, and therefore the weights of all points in a cluster which is a balancing supply system are equal. Moreover, the balancing condition implies that the weights of two distinct clusters must agree. Therefore there is a unique balancing along the ray ρ_i. ■

For a transverse very affine arrangement of lines and conics $\mathcal{B}$, a straightforward condition for all the maximal clusters of a conic C to be balancing supply systems, is simply that for any two other conics C' and C'', all possible lines between points in $C' \cap C$ and $C'' \cap C$ are part of the arrangement.

III.6 Cohomologically tropical arrangements

In this section, we study two properties of transverse very affine arrangements of curves. First, we show that under certain conditions, the complement of the arrangement is *wunderschön* in the sense of [AAPS23, Definition 1.2], which is primarily a restriction on the Mixed Hodge structure of the cohomology of this complement and of the involved curves. Next we use the study of arroid fans which are tropical homology manifold as discussed in Section III.5, to show that for certain arrangements, the complement is *cohomologically tropical* in the sense of [AAPS23, Definition 1.1], i.e. that one may compute all the cohomology of the complement using only the tropical variety.

Let $\mathcal{B}$ be a transverse very affine arrangement of curves in the plane $\mathbb{P}^2_{\mathbb{C}}$, and $X_{\mathcal{B}}$ its complement, which we identify as a subvariety of its intrinsic torus $\mathbf{T}_{X_{\mathcal{B}}}$. The associated arroid $\mathbf{A} = \mathbf{A}_{\mathcal{B}}$ then yields the tropicalization of $X_{\mathcal{B}} \subseteq \mathbf{T}_{X_{\mathcal{B}}}$ as the support of the arroid fan $\Sigma_{\mathbf{A}}$.

Proposition III.6.1. *Let $\mathcal{B}$ be a transverse very affine arrangement of non-singular rational curves in $\mathbb{P}^2_{\mathbb{C}}$ such that no intersection point of the arrangement contains exactly the same curves, then $X_{\mathcal{B}}$ is a wunderschön variety.*

Proof. Consider the toric variety $\mathbb{CP}_{\Sigma_{\mathbf{A}}}$ associated to the (unimodular) fan $\Sigma_{\mathbf{A}}$ and the closure $\overline{X}_{\mathcal{B}} \subseteq \mathbb{CP}_{\Sigma_{\mathbf{A}}}$. Each cone σ of $\Sigma_{\mathbf{A}}$ gives a torus orbit $\mathbf{T}^{\sigma} \subseteq \mathbb{CP}_{\Sigma_{\mathbf{A}}}$, and let $X^{\sigma}_{\mathcal{B}} \coloneqq \overline{X}_{\mathcal{B}} \cap \mathbf{T}^{\sigma}$. To show that $X_{\mathcal{B}}$ is wunderschön, we must first show that each $X^{\sigma}_{\mathcal{B}}$ is non-singular.

For the minimal cone $\mathbf{0}$ of $\Sigma_{\mathbf{A}}$, corresponding to the central vertex, $X^{\mathbf{0}}_{\mathcal{B}}$ is the complement of the curves of $\mathcal{B}$, hence is non-singular. For a ray ρ_i of $\Sigma_{\mathbf{A}}$ corresponding to a divisor $i \in \mathcal{A}$ of the arroid, the intersection $X^{\rho_i}_{\mathcal{B}}$ is the curve of $\mathcal{B}$ corresponding to i, which is non-singular. For a ray $\rho_{\mathbf{p}}$ corresponding to a point $\mathbf{p} \in \mathcal{P}$ of the arroid, the intersection $X^{\rho_\rho}_{\mathcal{B}}$ is the exceptional curve of the blow-up at the corresponding point, which is also non-singular. Moreover for each cone $\sigma_{i,\mathbf{p}} = \mathrm{cone}(e_i, e_{\mathbf{p}})$ of $\Sigma_{\mathbf{A}}$, since no intersection point of the arrangement contains exactly the same curves, the intersection $X^{\sigma_{i,\mathbf{p}}}_{\mathcal{B}}$ is the intersection of the exceptional curve corresponding to $\mathbf{p}$ with the curve corresponding to i, which is exactly one point, hence non-singular.

Next, to show that $X_{\mathcal{B}}$ is wunderschön, we must verify that for each $X^{\sigma}_{\mathcal{B}}$, the mixed Hodge structure on $H^k(X^{\sigma}_{\mathcal{B}})$ is pure of weight $2k$. This follows for the two-dimensional cones from the assumption on points, and follows for each one-dimensional cone as it corresponds to a punctured non-singular rational curve. Finally, for $X^{\mathbf{0}}_{\mathcal{B}} = X_{\mathcal{B}}$, we consider Deligne's weight spectral sequence for the mixed Hodge structure [Del71, §7]. All the homology of $\overline{X}_{\mathcal{B}}$ is generated by that of $\mathbb{P}^2_{\mathbb{C}}$ together with that of the exceptional divisors of the blown-up points. Therefore, there is no odd degree cohomology of $\overline{X}_{\mathcal{B}}$, and since all the boundary divisors are rational, all odd rows of the spectral sequence are identically zero. Then it remains to show that the cohomology of the following two complexes, corresponding to the only non-zero even rows of the weight spectral sequence, is concentrated on the left

$$0 \longrightarrow \oplus_p H^0(p) \longrightarrow \oplus_D H^2(D) \longrightarrow H^4(\overline{X}_{\mathcal{B}}) \longrightarrow 0,$$

$$0 \longrightarrow \oplus_D H^0(D) \longrightarrow H^2(\overline{X}_{\mathcal{B}}) \longrightarrow 0.$$

Here the D's are the components of the boundary divisors compactifying $X_{\mathcal{B}}$ to $X_{\mathcal{B}}$, and the p's correspond to their pairwise intersections. Since the dual graph of the compactifying divisor is connected, the top complex only has cohomology on the left by [Hac08, Theorem 3.1]. Moreover, the homology $H_2(\overline{X}_{\mathcal{B}})$ is generated by the homology of $\mathbb{P}^2_{\mathbb{C}}$ along with the homology of the exceptional divisors of the blown-up points [GH94, p. 474], so the map in the lower complex is surjective, and gives the cohomology $\mathrm{Gr}_2^W H^2(X_{\mathcal{B}}) = 0$. We therefore have $\mathrm{Gr}_i^W H^k(X_{\mathcal{B}})$ is 0 if $i \neq 2k$, i.e. the mixed Hodge structure is pure of weight $2k$. ■

A transverse very affine arrangements $\mathcal{B}$ of non-singular rational curves in $\mathbb{P}^2_{\mathbb{C}}$, i.e. of lines and conics, such that that no intersection point of the arrangement contains exactly the same curves, will be called *simple*. For simple arrangements, we relate the rational cohomology of their complements with the tropical cohomology of their tropicalizations, using the notion of cohomologically tropical varieties.

We give a brief description of the *cohomologically tropical* property for varieties and their tropicalizations, referring to [AAPS23] for the original definition and greater details. For $X \subseteq \mathbf{T}$ a subvariety of a torus, and $\mathrm{Trop}(X)$ its tropicalization, picking a unimodular fan Σ supported on $\mathrm{Trop}(X)$ yields a compactification $\overline{X}$ of X inside the toric variety $\mathbb{CP}_{\Sigma}$, as well as a compactification $\overline{\mathrm{Trop}(X)}$ of $\mathrm{Trop}(X)$ inside the tropical toric variety $\mathbb{TP}_{\Sigma}$. There is a map $\tau^*\colon H^\bullet(\overline{\mathrm{Trop}(X)}) \to H^\bullet(\overline{X})$, relating the $\mathbb{Q}$-coefficient tropical cohomology of $\overline{\mathrm{Trop}(X)}$ to the rational cohomology ring of $\overline{X}$ (see [AAPS23] for more on this map). This map can also be defined for each of the reduced stars and the corresponding projections of the variety X. If all these maps are isomorphisms, X is said to be *cohomologically tropical*. For simple arrangements, we have the following result regarding which have cohomologically tropical complements.

Theorem III.6.2. *Let $X_{\mathcal{B}}$ be the complement of a simple arrangement $\mathcal{B}$. Then $X_{\mathcal{B}}$ is cohomologically tropical if and only if the corresponding arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$ is uniquely balanced along each of its rays.*

Proof. By Proposition III.6.1, this complement is necessarily wunderschön. Therefore, the equivalence described in [AAPS23, Theorem 6.1] reduces to an equivalence between $X_{\mathcal{B}}$ being cohomologically tropical and $\mathrm{trop}(X_{\mathcal{B}}) = \Sigma_{\mathbf{A}_{\mathcal{B}}}$ being a tropical homology manifold. The result then follows from Theorem III.5.4. ■

III.7 Maximal subvarieties

In this section, we will study the question of maximality for arrangements of curves using the arroid abstraction we have developed so far. Let X be a complex subvariety of $\mathbb{P}^2_{\mathbb{C}}$ defined over $\mathbb{R}$, with $X(\mathbb{R})$ its set of real points and $X(\mathbb{C})$ its set of complex points. The *Smith-Thom* inequality gives bounds for the sum of the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers b_k as follows,

$$b_\bullet(X(\mathbb{R})) := \sum_{k\geq 0} b_k(X(\mathbb{R})) \leq \sum_{k\geq 0} b_k(X(\mathbb{C})) =: b_\bullet(X(\mathbb{C})).$$

The variety X is said to be *maximal* if equality holds. For a simple arrangement $\mathcal{B}$ of curves, i.e. consisting of lines and conics, defined over the reals, with all intersections being real, we will now use tropical homology manifolds to give a condition for maximality of the complement of the arrangement. Along with the two next lemmas, this will give examples of varieties satisfying conditions (a), (b) and (c) of [AM22, p. 3], as described in the introduction.

Lemma III.7.1. *Let $\mathcal{B}$ be an arrangement of curves containing at least one line in $\mathbb{P}^2_{\mathbb{C}}$, and $X_{\mathcal{B}}$ its complement. Then the homology groups $H_1(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z})$ and $H_2(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z})$ are torsion-free, and $H_k(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z}) = 0$ for $k > 2$.*

Proof. Since the arrangement contains at least one line, it is affine, therefore the homology group $H_1(X_{\mathcal{B}})$ is torsion-free, see e.g. [Dim92, Corollary 4.1.4]. Adapting similar methods, one may consider the long exact sequence in homology for the pair $(\mathbb{P}^2_{\mathbb{C}}, X_{\mathcal{B}})$, which yields

$$\cdots \to H_3(\mathbb{P}^2_{\mathbb{C}};\mathbb{Z}) \to H_3(\mathbb{P}^2_{\mathbb{C}}, X_{\mathcal{B}}(\mathbb{C});\mathbb{Z}) \to H_2(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z}) \to H_2(\mathbb{P}^2_{\mathbb{C}};\mathbb{Z}) \to \cdots$$

Since $H_3(\mathbb{P}^2_{\mathbb{C}};\mathbb{Z}) = 0$ and $H_2(\mathbb{P}^2_{\mathbb{C}};\mathbb{Z}) = \mathbb{Z}$, it suffices to show that $H_3(\mathbb{P}^2_{\mathbb{C}}, X_{\mathcal{B}}(\mathbb{C});\mathbb{Z})$ is torsion-free. Let T be a closed tubular neighborhood of the curve $C = \cup_{C\in\mathcal{B}} C$, with boundary ∂T. By excision, $H_3(\mathbb{P}^2_{\mathbb{C}}, X_{\mathcal{B}}(\mathbb{C});\mathbb{Z}) \cong H_3(T, \partial T;\mathbb{Z})$, and by Lefschetz duality [Hat02, p. 254], $H_3(T, \partial T;\mathbb{Z}) \cong H^1(T;\mathbb{Z})$. This last group is torsion-free since H^1-cohomology is always torsion-free.

Moreover, since there is at least one line in the arrangement, the variety $X_{\mathcal{B}}(\mathbb{C})$ is affine. Therefore all homology $H_k(X_{\mathcal{B}}(\mathbb{C});\mathbb{Z})$ vanishes for $k > 2$ by [Mil63, Theorem 7.1]. ■

Lemma III.7.2. *Let $\mathcal{B}$ be a simple arrangement in $\mathbb{P}^2_{\mathbb{C}}$, with all the curves defined over the reals, and all intersection points being real. Then $H_1(X_{\mathcal{B}}(\mathbb{R});\mathbb{Z})$ is trivial.*

Proof. This follows by induction. By adding the curve C to the arrangement $X_{\mathcal{B}\smallsetminus\{C\}}$, the real part $C(\mathbb{R})$ is an S^1 not contained in a component of $X_{\mathcal{B}\smallsetminus\{C\}}(\mathbb{R})$ since all intersection points are real, so no H_1 homology is added. ■

Theorem III.7.3. *Let $\mathcal{B}$ be a simple arrangement in $\mathbb{P}^2_{\mathbb{C}}$, with all the curves defined over the reals. Suppose that all intersection points in the arrangement are real and* $\mathrm{Trop}(X_{\mathcal{B}})$ *is a tropical homology manifold. Then the complement $X_{\mathcal{B}}$ is maximal.*

Proof. The arrangement $\mathcal{B}$ consists of smooth rational curves in $\mathbb{P}^2_{\mathbb{C}}$, of which at least three are lines, so we first consider the smaller arrangement $\mathcal{B}_L$ consisting of only the lines. One may prove that $\mathcal{B}_L$ is maximal, see [OT92, Corollary 5.95], so we may proceed by induction on the number of curves in the arrangement.

Let $\mathcal{B}$ be a simple arrangement of curves, with all intersection points being real, and with the corresponding arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$ being a tropical homology manifold. By Proposition III.6.1, the arrangement complement $X_{\mathcal{B}}$ is wunderschön, and so by [AAPS23, Theorem 6.1], it is cohomologically tropical. By [AAPS23, Theorem 6.2], we have isomorphisms $H^k(X_{\mathcal{B}}(\mathbb{C})) \cong H^{k,0}(\Sigma_{\mathbf{A}_{\mathcal{B}}})$, and in particular $b_k(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Z}/2\mathbb{Z}) = \dim\ H^{k,0}(\Sigma_{\mathbf{A}_{\mathcal{B}}})$, for all k since the homology is torsion-free by Lemma III.7.1.

For any choice of curve $C \in \mathcal{B}$ and corresponding $i \in \mathcal{A}$, the reduced star fan of the ray ρ_i is given by $\Sigma_{\mathbf{A}/i}$ by Proposition III.4.8, and is a tropical homology manifold by assumption. It follows from Proposition III.6.1 that the punctured curve C^* tropicalizing to $\Sigma_{\mathbf{A}/i}$ is wunderschön, and so by [AAPS23, Theorem 6.2], $\dim H^k(C^*) = \dim H^{k,0}(\Sigma_{\mathbf{A}/i})$ for all k. Moreover, $C^*(\mathbb{C})$ is homotopic to a wedge of circles, so that its homology is torsion-free, hence $b_k(C^*(\mathbb{C})) = b_k(C^*(\mathbb{C}); \mathbb{Z}/2\mathbb{Z}) = \dim\ H^{k,0}(\Sigma_{\mathbf{A}/i})$ for all k.

Similarly, we study the arrangement $\mathcal{B} \setminus \{C\}$, corresponding to the contracted arroid $\mathbf{A}_{\mathcal{B}\setminus i}$. The tropicalization of its complement $X_{\mathcal{B}\setminus\{C\}}$ is the arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}\setminus i}}$, which by Lemma III.5.5 is a tropical homology manifold. By induction assumption, the complement $X_{\mathcal{B}\setminus\{C\}}$ is maximal, i.e. $b_\bullet(X_{\mathcal{B}\setminus\{C\}}(\mathbb{R})) = b_\bullet(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C}))$. By Proposition III.6.1, $X_{\mathcal{B}\setminus\{C\}}$ is wunderschön, so that by [AAPS23, Theorem 6.1] it is cohomologically tropical, and therefore $H^{k,0}(\Sigma_{\mathbf{A}_{\mathcal{B}\setminus i}}) \cong H^k(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C}))$ for all k by [AAPS23, Theorem 6.2]. Therefore $b_k(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C})) = b_k(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C}); \mathbb{Z}/2\mathbb{Z}) = \dim\ H^{k,0}(\Sigma_{\mathbf{A}_{\mathcal{B}\setminus i}})$ for all k.

By Lemma III.7.2, $b_\bullet(X_{\mathcal{B}}(\mathbb{R})) = b_0(X_{\mathcal{B}}(\mathbb{R}))$ for any simple arrangement with real intersection points. Moreover, each arc of $C^*(\mathbb{R})$ increases the number of connected components of $X_{\mathcal{B}\setminus\{C\}}(\mathbb{R})$ by 1, so counting the number of connected components of $X_{\mathcal{B}}(\mathbb{R})$, we have $b_0(X_{\mathcal{B}}(\mathbb{R})) = b_0(X_{\mathcal{B}\setminus\{C\}}(\mathbb{R})) + b_0(C^*(\mathbb{R}))$. Note also that the number of punctures of $S^1 \cong C(\mathbb{R})$ by real points is $b_0(C^*(\mathbb{R}))$, so $C^*(\mathbb{C})$ is a punctured Riemann sphere, which gives $b_0(C^*(\mathbb{R})) = b_\bullet(C^*(\mathbb{C}))$. Summarizing, we have

$$\begin{aligned} b_\bullet(X_{\mathcal{B}}(\mathbb{R})) = b_0(X_{\mathcal{B}}(\mathbb{R})) &= b_0(X_{\mathcal{B}\setminus\{C\}}(\mathbb{R})) + b_0(C^*(\mathbb{R})) \\ &= b_\bullet(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C})) + b_\bullet(C^*(\mathbb{C})). \end{aligned}$$

By Lemma III.5.2, we have $\dim H^{k-1,0}(\Sigma_{\mathbf{A}/i}) + \dim H^{k,0}(\Sigma_{\mathbf{A}\setminus i}) = \dim H^{k,0}(\Sigma_{\mathbf{A}})$ for all k, which when combined with the equalities between tropical cohomology and singular cohomology described above, yield $b_\bullet(X_{\mathcal{B}\setminus\{C\}}(\mathbb{C})) + b_\bullet(C^*(\mathbb{C})) = b_\bullet(X_{\mathcal{B}}(\mathbb{C}))$. ■

Example III.7.4. We illustrate Theorem III.7.3 by the example displayed in Figure III.6. It is constructed by selecting four points, drawing all lines between them, and then selecting any number of non-singular conics passing through those same four points. To see that these arrangements are tropical homology

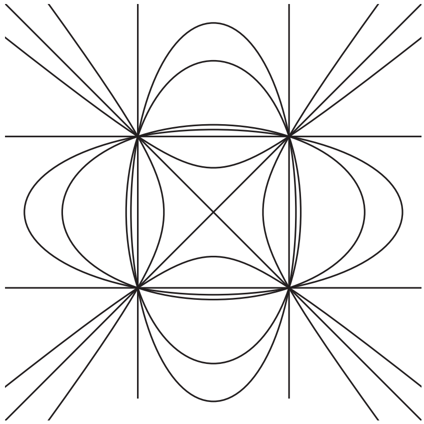

Figure III.6: A maximal arrangement

manifolds, it suffices to note that in each intersection point of a line with a conic, multiple other lines pass through the same point, so that all rays associated to lines of the arroid fan are uniquely balanced by Lemma III.5.8, and all intersection points of any pair of conics are connected by lines, so that by Lemma III.5.11 all rays associated to conics are uniquely balanced. By Theorem III.5.4, the arroid fan is therefore a tropical homology manifold, hence by Theorem III.7.3 each such arrangement is maximal.

Finally, simple real arrangements of curves, all intersecting in real points, with the tropicalization of the complement being a tropical homology manifold, satisfy conditions (a), (b) and (c) of [AM22, p. 3].

Theorem III.7.5. *Let $\mathcal{B}$ be a simple arrangement of real curves in $\mathbb{P}^2_{\mathbb{C}}$, with all intersection points being real, and such that the tropicalization* $\mathrm{Trop}(X_{\mathcal{B}})$*, which is supported on the arroid fan $\Sigma_{\mathbf{A}_{\mathcal{B}}}$, is a tropical homology manifold. Then the following four properties are satisfied:*

(a) $H^i(X_{\mathcal{B}}(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) = 0$ *for* $i \geq 1$,

(b) $X_{\mathcal{B}}$ *is a maximal variety,*

(c) *the mixed Hodge structure on* $H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Q})$ *is pure of type* (i, i) *and* $H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Z})$ *is torsion-free for* $i \geq 1$*, and*

(d) $\dim_{\mathbb{Q}} H^i(X_{\mathcal{B}}(\mathbb{C}); \mathbb{Q}) = \sum_j \dim_{\mathbb{Q}} H^{i,j}(\Sigma_{\mathbf{A}_{\mathcal{B}}})$ *for each* $i \geq 0$.

Proof. The first property follows from Lemma III.7.2, the second from Theorem III.7.3. The purity of the mixed Hodge structure on rational cohomology is satisfied because $X_{\mathcal{B}}$ is wunderschön by Proposition III.6.1, and the integer homology is torsion-free by Lemma III.7.1. The last property follows from Theorem III.6.2, as $X_{\mathcal{B}}$ is cohomologically tropical since the tropicalization is a tropical homology manifold. ■

In particular, the family of arrangements described in Example III.7.4 gives examples of the varieties satisfying the conditions of [AM22].

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