Tropical Poincaré Duality Spaces

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LAGARTOS, 2022-03-25

Appetizer

[Itenberg-Katzarkov-Mikhalkin-Zharkov 19]

X tropical variety $\rightsquigarrow$ tropical homology $H_{p,q}(X)$ $\quad 0 \le p,q \le d$
dim $= d$ $\qquad$ —— cohomology $H^{p,q}(X)$

Balancing condition $\rightsquigarrow$ fundamental class $[X] \in H_{d,d}(X)$

$[X]$ induces a cap product [Jell-Shaw-Smacka ; Jell-Rau-Shaw; Gross-Shokrieh; Amini-Piquerez]

$$\frown [X] : H^{p,q}(X) \longrightarrow H_{d-p,d-q}(X).$$

When all are isomorphisms, say X has tropical Poincaré duality (or is a TPD space)

Example: Smooth tropical varieties (built out of Bergman fans of matroids)

Which tropical varieties have TPD? $\rightsquigarrow$ Which fans have TPD?

Glueing, Mayer-Vietoris
[JSS; JRS; GS; AP]

For fans with TPD at all faces (tropically smooth in [AP]):

Theorem [AP]: Chow ring $A^\bullet(\Sigma_M) \cong H^{2\bullet}(\overline{\Sigma})$ total trop. cohom.

[AP]: "Kähler package" for tropical varieties built from such fans.

Theorem links tropical homology to Chow rings of toric varieties and matroids via work of [Adiprasito-Huh-Katz].

Plan.

1. Fans, tropical (co)homology, balancing & cap product
2. Tropical Poincaré duality
3. TPD from faces and Local TPD

Fans

$$\Sigma = \{ \overset{\text{cone}}{\sigma} \subset \overset{\mathbb{Z}^n}{\underset{}{N}} \} \text{ finite.}$$

($N \cong \mathbb{Z}^n$)

such that

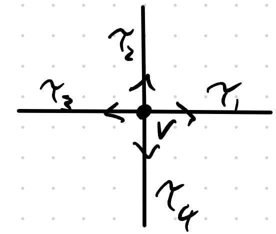
1. $\sigma \in \Sigma$ & τ face of σ $(\tau \leq \sigma)$ then $\tau \in \Sigma$
2. $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 \in \Sigma$

Write $\Sigma^i = \{ \sigma \in \Sigma \mid \dim \sigma = i \}$

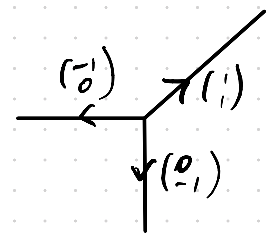
All fans are assumed to be pure dimensional.

Examples

- "The cross"



- Tropical line



- The Bergman fan of a matroid [Sturmfels; Ardila-Klivans; Feichtner-Sturmfels]

Tropical (co)homology [IKMZ]

The <u>p-th multi-tangent cosheaf</u> $\mathcal{F}_p^{\mathbb{Z}}(\sigma)$, $p=0,\dots,d$, is given by:

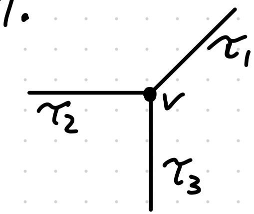
- $\mathcal{F}_p^{\mathbb{Z}}(\sigma) := \sum_{\sigma \leq \gamma} \Lambda^p L_{\mathbb{Z}}(\gamma) \subseteq \Lambda^p N$
- $\tau \leq \sigma \rightsquigarrow c_{\sigma,\tau}: \mathcal{F}_p^{\mathbb{Z}}(\sigma) \to \mathcal{F}_p^{\mathbb{Z}}(\tau)$ is the subspace inclusion.

Dualizing $\rightsquigarrow$ <u>p-th multi-tangent sheaf</u> $\mathcal{F}^p_{\mathbb{Z}}$

Tensoring with $\mathbb{R}$ $\rightsquigarrow$ $\mathcal{F}_p^{\mathbb{R}}$, $\mathcal{F}^p_{\mathbb{R}}$

Example

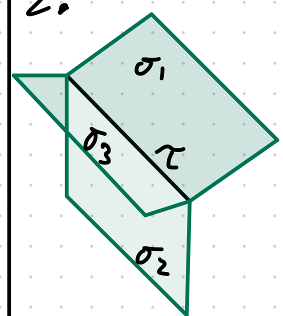
1.



$$\mathcal{F}_1^{\mathbb{Z}}(\tau_1) = L_{\mathbb{Z}}(\tau_1) = \langle \begin{pmatrix}1\\1\end{pmatrix} \rangle \subseteq N$$

$$\mathcal{F}_1^{\mathbb{Z}}(v) = L_{\mathbb{Z}}(\tau_1) + L_{\mathbb{Z}}(\tau_2) + L_{\mathbb{Z}}(\tau_3)$$
$$= \langle \begin{pmatrix}1\\1\end{pmatrix} \rangle + \langle \begin{pmatrix}-1\\0\end{pmatrix} \rangle + \langle \begin{pmatrix}0\\-1\end{pmatrix} \rangle$$
$$\cong \mathbb{Z}^2 = N$$

2.



$\subset \mathbb{Z}^3$

$$\mathcal{F}_1^{\mathbb{Z}}(\tau) \cong \mathbb{Z}^3$$

$$\mathcal{F}_2^{\mathbb{Z}}(\tau) = \sum_{i=1}^{3} \Lambda^2 L_{\mathbb{Z}}(\sigma_i) \ncong \Lambda^2 \mathbb{Z}^3$$

Tropical (co)homology has many equivalent definitions:

- simplicial [Mikhalkin-Zharkov],
- superforms [JSS]
- sheaf theoretic [MZ; GS]
- local coefficients [IKMZ; MZ].

Here we use the cellular version [IKMZ]:

X tropical variety with cellular decomp. has cochain complexes:

$$0 \to \bigoplus_{v \in X^0} \mathcal{F}^p(v) \xrightarrow{d} \bigoplus_{\substack{e \in X^1 \\ e \text{ compact}}} \mathcal{F}^p(e) \xrightarrow{d} \bigoplus_{\substack{\sigma \in X^2 \\ \sigma \text{ compact}}} \mathcal{F}^p(\sigma) \to \dots$$

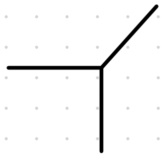
$\rightsquigarrow$ Tropical cohomology $H^{p,q}(X)$

Remark: The tropical cohomology groups are invariant under subdivisions. (if sufficiently fine, need pointed).

Proposition: For tropical fans.

$$H^{p,q}(\Sigma) = \begin{cases} \mathcal{F}^p_R(v) & \text{for } q=0 \\ 0 & \text{otherwise} \end{cases}$$

Example:



$H^{0,0}(\Sigma) = \mathbb{Z}$ $\quad H^{0,1}(\Sigma) = 0$

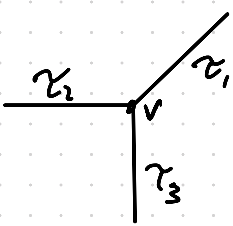
$H^{1,0}(\Sigma) = \mathcal{F}^1_{\mathbb{Z}}(v) = \mathbb{Z}^2$ $\quad H^{1,1}(\Sigma) = 0$

Tropical Borel-Moore chain complexes $C_\bullet^{BM}(\Sigma, \mathcal{F}_p^R)$:

$$0 \to \bigoplus_{\sigma \in \Sigma^d} \mathcal{F}_p^R(\sigma) \xrightarrow{\partial} \bigoplus_{\tau \in \Sigma^{d-1}} \mathcal{F}_p^R(\tau) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigoplus_{e \in \Sigma^1} \mathcal{F}_p^R(e) \xrightarrow{\partial} \mathcal{F}_p^R(v) \to 0$$

$\to$ <u>Tropical Borel-Moore homology</u> $H_{p,q}^{BM}(\Sigma) := H_q(C_\bullet^{BM}(\Sigma, \mathcal{F}_p^R))$

<u>Example</u>



$$\mathcal{F}_0^{\mathbb{Z}}: \quad 0 \to \overset{\mathbb{Z}}{\overset{\|}{\mathcal{F}_0^{\mathbb{Z}}(\tau_1)}} \oplus \overset{\mathbb{Z}}{\overset{\|}{\mathcal{F}_0^{\mathbb{Z}}(\tau_2)}} \oplus \overset{\mathbb{Z}}{\overset{\|}{\mathcal{F}_0^{\mathbb{Z}}(\tau_3)}} \xrightarrow{(1\ 1\ 1)} \overset{\mathbb{Z}}{\overset{\|}{\mathcal{F}_0^{\mathbb{Z}}(v)}} \to 0$$

$$\rightsquigarrow H_{0,0}^{BM}(\Sigma) = 0 \qquad H_{0,1}^{BM}(\Sigma) = \mathbb{Z}^2$$

$$\mathcal{F}_1^{\mathbb{Z}}: \quad 0 \to \underset{\langle \binom{1}{1} \rangle}{\underset{\|}{\mathcal{F}_1^{\mathbb{Z}}(\tau_1)}} \oplus \underset{\langle \binom{-1}{0} \rangle}{\underset{\|}{\mathcal{F}_1^{\mathbb{Z}}(\tau_2)}} \oplus \underset{\langle \binom{0}{-1} \rangle}{\underset{\|}{\mathcal{F}_1^{\mathbb{Z}}(\tau_3)}} \xrightarrow{(\mathrm{id}\ \mathrm{id}\ \mathrm{id})} \mathcal{F}_1^{\mathbb{Z}}(v) \to 0$$

$$\mathcal{F}_1^{\mathbb{Z}}(v) = \langle \tbinom{-1}{0}, \tbinom{1}{1}, \tbinom{0}{-1} \rangle = \langle \tbinom{1}{0}, \tbinom{0}{1} \rangle$$

$$\rightsquigarrow H_{1,0}^{BM}(\Sigma) = 0 \quad \& \quad \binom{1}{1} + \binom{-1}{0} + \binom{0}{-1} = 0 \Rightarrow H_{1,1}^{BM}(\Sigma) = \mathbb{Z}$$

Balancing condition

A fan Σ of dim d is balanced if $\forall \tau \in \Sigma^{d-1}$ there is a balancing:

$$\sum_{\tau < \sigma} w_\sigma v_{\sigma/\tau} = 0$$

using a given weight function $w : \Sigma^d \to R$.
with no zero divisors

Proposition Each balancing of a fan gives a class $\langle [\Sigma, w] \rangle \subseteq H^{BM}_{d,d}(\Sigma)$.

Proof sketch: $\Lambda^{d-1} L_{\mathbb{Z}}(\tau) = \langle \Lambda_\tau \rangle$

$\sigma > \tau$: $\mathcal{F}_d(\sigma) = \langle \Lambda_\tau \wedge v_{\sigma/\tau} \rangle$

$$\partial\left(\left(w_\sigma \Lambda_\tau \wedge v_{\sigma,\tau} \right)_{\sigma \in \Sigma^d} \right)_\tau = \Lambda_\tau \wedge \left(\sum_{\tau < \sigma} w_\sigma v_{\sigma,\tau} \right)$$

Tropical cap product [JRS 18]

$$\frown [\Sigma, w] : H^{p,q}(\Sigma) \to H^{BM}_{d-p,d-q}(\Sigma)$$

Covector contraction:

$$\mathcal{F}^1(\sigma) \times \mathcal{F}_d(\sigma) \longrightarrow \mathcal{F}_{d-1}(\sigma)$$

$$(f, \underset{\substack{\| \\ v_1 \wedge \cdots \wedge v_d}}{\Lambda_\sigma}) \longmapsto \sum f(v_i) v_1 \wedge \cdots \wedge \hat{v_i} \wedge \cdots \wedge v_d$$

Using [Bourbaki Algebra III.11.9], get multilinear maps:

$$\lrcorner : \mathcal{F}^p(\sigma) \times \mathcal{F}_d(\sigma) \to \mathcal{F}_{d-p}(\sigma)$$

Moreover: $\gamma \in \Sigma^q$, $\tau \in \Sigma^{d-q}$, $\gamma, \tau \leq \sigma \in \Sigma^d$,

$$\lrcorner : \mathcal{F}^p(\gamma) \times \mathcal{F}_d(\sigma) \longrightarrow \mathcal{F}_{d-p}(\tau)$$

$$(v, \Lambda_\sigma) \longmapsto \iota_{\sigma,\tau}\left(\rho_{\gamma,\sigma}(v) \lrcorner \Lambda_\sigma \right)$$

Tropical cap product (cont.)

$\gamma, \tau < \sigma \in \Sigma^d$, $\gamma \in \Sigma^q$, $\tau \in \Sigma^{d-q}$:

$$\lrcorner : \mathcal{F}^p(\gamma) \times \mathcal{F}_d(\sigma) \to \mathcal{F}_{d-p}(\tau)$$

gives maps in cohomology:

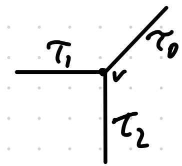
$$\frown [\Sigma, \omega] : H^{p,q}(\Sigma) \to H^{BM}_{d-p,d-q}(\Sigma)$$

For fans, $H^{p,q} = 0$ for $q \neq 0$, only non-zero maps are:

$$\frown [\Sigma, \omega] : \underset{\substack{\| \\ \mathcal{F}_R^p(v)}}{H^{p,0}(\Sigma)} \to H^{BM}_{d-p,d}(\Sigma)$$

Proposition [A] Σ fan $\Rightarrow \frown [\Sigma, \omega]$ injective.

Example



$H^{0,0} = \mathbb{Z}$ $\qquad H^{BM}_{0,1} = \mathbb{Z}^2$

$H^{1,0} = \mathbb{Z}^2$ $\qquad H^{BM}_{1,1} = \mathbb{Z}$

$$\begin{array}{ccc}
\frown [\Sigma, \omega] : \mathcal{F}_R^1(v) & \longrightarrow & H^{BM}_{0,1}(\Sigma) \\
\| & & \| \\
\left(\sum_\tau L(\tau)\right)^\vee & & \ker\left(\bigoplus \mathbb{Z}\tau_i \xrightarrow{(1,1,1)} \mathbb{Z}v\right) \\
\psi & & \\
\gamma & \longmapsto & \left(\rho_{v,\tau_i}(\gamma) \lrcorner 1_{\tau_i}\right)_i
\end{array}$$

$$e_1^* \frown [\Sigma, \omega] = \left(e_1^*(e_1 + e_2), e_1^*(-e_1), e_1^*(-e_2)\right) = (1, -1, 0)$$

$$e_2^* \frown [\Sigma, \omega] = (1, 0, -1)$$

2. Tropical Poincaré Duality.

Tropical Poincaré Duality.

A fan satisfies tropical Poincaré duality if

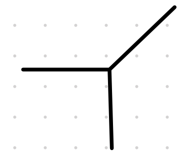
$$\frown [\Sigma, w] : H^{p,q}(\Sigma) \longrightarrow H^{BM}_{d-p,d-q}(\Sigma)$$

are isomorphisms for all $p,q = 0, \dots, d$.

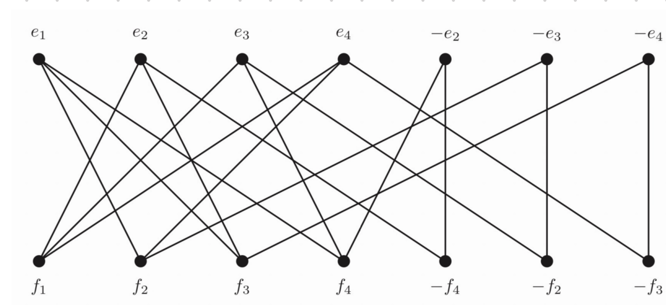
Sufficient to check:

- $H^{BM}_{d-p,d-q}(\Sigma) = 0$ for $q \neq 0$
- $\frown [\Sigma, w] : H^{p,0}(\Sigma) \longrightarrow H^{BM}_{d-p,d}(\Sigma)$ is surjective

Example

- Tropical line 

- Bergman fans of matroids satisfy TPD over $\mathbb{R}$ [Jell-Shaw-Smacka 19] and over $\mathbb{Z}$ [Jell-Rau-Shaw 18; Gross-Shokrieh].

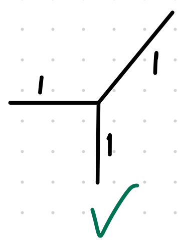
- 

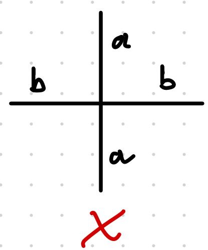
[Babaee-Huh 17] Not matroidal.

Classification in dimension one

Proposition[A] A one-dimensional balanced (Σ, w) is a TPD space iff it is uniquely balanced with unit weights.

Examples





Unique Balancing $\iff$ $H^{BM}_{1,1}(\Sigma) = \langle [\Sigma, w] \rangle \cong R$

Proof sketch:

- $H^{BM}_{1-p,0}(\Sigma) = 0$ for $p = 0, 1$ is automatic.
- Suffices to check surjectivity of
  - $\frown [\Sigma, w] : H^{0,0} \longrightarrow H^{BM}_{1,1}$, where $H^{0,0} = \mathcal{F}^0(v) = R$ and $H^{BM}_{1,1} = \langle [\Sigma, w] \rangle \cong R$

    is scalar multiplication.
  - $\frown [\Sigma, w] : \mathcal{F}^1(v) \longrightarrow H^{BM}_{0,1} = \ker\left(R^{|\Sigma^1|} \xrightarrow{(1,\dots,1)} R\right)$

  Strategy:

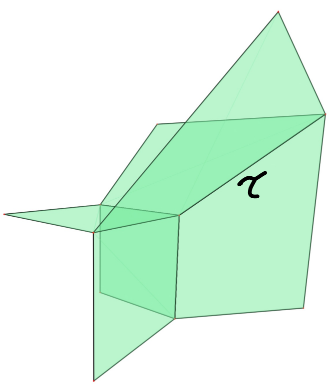
  a. Find a nice basis for $H^{BM}_{0,1}$

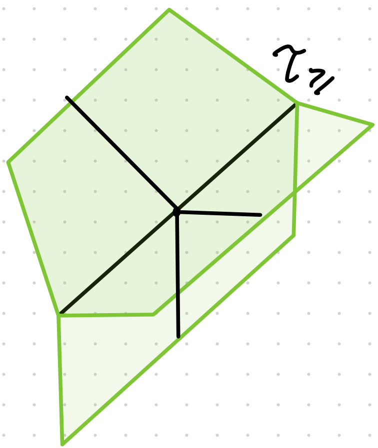
  b. Find nice covectors contracting on $[\Sigma, w]$ to hit exactly the basis from a.

3. TPD from faces and Local TPD

Stars

The <u>star</u> $\gamma_{\geq}$ at a face $\gamma \in \Sigma$ is the fan with support
$$\bigcup_{\gamma \leq k} \left(\mathbb{Z}_+ k + \mathbb{Z}_+(-k)\right)$$





(For tropical cohom. $\longrightarrow$ subdivide appropriately)

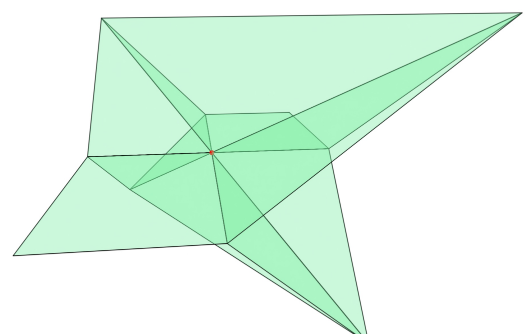
Theorem [A]

(Σ, w) dim $d \geq 2$ balanced fan, with $H^{BM}_{p,q}(\Sigma) = 0$ for $q \neq d$, $\forall p$.

If $\gamma_{\geq}$ satisfies TPD $\forall \gamma \neq$ vertex.

then (Σ, w) satisfies TPD.

Note: Vanishing assumption cannot be dropped:



[A18]

Sketch of the proof

1. Construct a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_R^p(v) \arrow[r, "\delta^0"] \arrow[d, "\frown[\Sigma,w]"] & \bigoplus_{\tau\in\Sigma^1} \mathcal{F}_R^p(\tau) \arrow[r, "\delta^1"] \arrow[d, "\oplus_\tau \frown[\tau_{\succeq},w]"] & \bigoplus_{\sigma\in\Sigma^2} \mathcal{F}_R^p(\sigma) \arrow[r, "\delta^2"] \arrow[d, "\oplus_\sigma \frown[\sigma_{\succeq},w]"] & \cdots \\
0 \arrow[r] & H^{BM}_{d,d-p}(\Sigma;R) \arrow[r, "\oplus_\alpha \overline{d^0_\alpha}"] & \bigoplus_{\tau\in\Sigma^1} H^{BM}_{d,d-p}(\tau_{\succeq};R) \arrow[r, "\oplus_\alpha \overline{d^1_\alpha}"] & \bigoplus_{\sigma\in\Sigma^2} H^{BM}_{d,d-p}(\sigma_{\succeq};R) \arrow[r, "\oplus_\alpha \overline{d^2_\alpha}"] & \cdots
\end{tikzcd}
$$

2. Prove exactness in the lower row, using the spectral sequence associated to a double complex.

3. TPD on faces $\Rightarrow$ vertical isomorphisms $\Rightarrow$ $\frown[\Sigma, w]$ iso.

Local TPD spaces

A fan Σ is a local TPD space (or tropically smooth [AP]) if $\forall \gamma \in \Sigma$, the star $\gamma_{\geq}$ satisfies TPD.

Known examples:
- Bergman fans of matroids
- Smoothness is shellable [AP, Theorem 10.4]

Theorem [A;AP] Tropical varieties built from local TPD spaces satisfy TPD.

Proof: Same protocol as [JSS;JRS]

Theorem [AP] Σ saturated unimodular + local TPD $\Longrightarrow A^\bullet(\Sigma) \xrightarrow{\cong} H^{2\bullet}(\overline{\Sigma})$.

Theorem [AP] (Tropical Deligne resolution) Σ unimodular + local TPD. $\Longrightarrow \exists$ L.E.S.

$$0 \to \mathcal{F}^p(0) \to \bigoplus_{\sigma \in \Sigma^p} H^0(\overline{\sigma_{\geq}}) \to \bigoplus_{\gamma \in \Sigma^{p-1}} H^2(\overline{\gamma_{\geq}}) \to \cdots \to \bigoplus_{e \in \Sigma^1} H^{2p-2}(\overline{e_{\geq}}) \to H^{2p}(\overline{\Sigma}) \to 0.$$

Can we classify local TPD spaces?

Theorem [A]

R a PID, Σ dim d R-balanced. Then Σ is a local TPD space iff:

- $H^{BM}_{p,q}(\gamma_{\geq}; R) = 0 \quad \forall \gamma \in \Sigma$ & $q \neq d$, and
- $\forall \beta$ of codim 1, $\beta_{\geq}$ is a TPD space

Proof sketch: $(\Rightarrow)$ By definition

$(\Leftarrow)$ Recursively apply the previous theorem

Remark: Classification in dim 1 [A] + tropical Künneth formula [GS]

$\Rightarrow$ Only need "unique balancing" of codim 1 faces

Where to go next?

Question (Geometry of BM homology vanishing).

Let (Σ, w) be an R-balanced d-dimensional fan. Can the fans with $H^{BM}_{p,q}(\sigma_{\geq}) = 0 \ \forall \sigma, q \neq d$, be geometrically characterized?

Question (Global vs local TPD)

Let (Σ, w) be an fan which satisfies TPD over R.
Is it also locally TPD?

Thank you!