2022-05-20

K3 Seminar

Grothendieck and Leray spectral sequence.

1. Overview (In particular ~~pre...~~ details in [Weibel]).

2. ~~Derived~~ functors

$\mathcal{A}, \mathcal{B}$ abelian categories, $F: \mathcal{A} \to \mathcal{B}$ left exact ($0 \to A' \to A \to A'' \to 0$ exact $\Rightarrow 0 \to FA' \to FA \to FA''$.)

$\mathcal{A}$ has enough injectives i.e. $\forall A \in \mathcal{A}$, $\exists$ injective object $I \in \mathcal{A}$ and an injection $A \hookrightarrow I$.

$$\hookrightarrow \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\exists \beta"] \\ & I & \end{tikzcd}$$

Define the Right derived functors of F to be given by $R^iF(A) := H^i(F(I^\circ))$, where $A \hookrightarrow I^\circ$ an injective resolution

Theorem: These exist, unique up to isomorphism, and short exact sequences in $\mathcal{A}$ induce exact complexes in $\mathcal{B}$.

Can we do this for complexes?

Let $A^\circ \in Ch_{\geq}(\mathcal{A})$ be a (bounded below) (co)chain complex. Then, for I° a complex of injectives with a (co)chain map $i^\circ: M^\circ \to I^\circ$ which is a quasi-iso, i^k injective $\forall k$, we set $R^iF(M^\circ) := H^i(F(I^\circ))$. (Note that if $M^\circ = A[0]$, this is the same def as before).

Claim: Such an injective resolution exists.

3. Horseshoe Lemma and Cartan-Eilenberg resolution

Horseshoe Lemma $\mathcal{A}$ abelian w/ enough injectives,

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ 0 \arrow[r] & B \arrow[r] \arrow[d] & I_B^0 \arrow[r] & I_B^1 \arrow[r] & I_B^2 \cdots \\ & A \arrow[d] & & & \\ 0 \arrow[r] & C \arrow[r] \arrow[d] & I_C^0 \arrow[r] & I_C^1 \arrow[r] & I_C^2 \cdots \\ & 0 & & & \end{tikzcd}$$

column exact and rows are injective resolutions.

Set $I_A^n = I_B^n \oplus I_C^n$, then the I_A^n form an injective resolution of A, and the column lifts to an exact sequence of complexes $0 \to I_B \xrightarrow{i} I_A \xrightarrow{\pi} I_C \to 0$ i inclusion and π projection.

Def A (right) Cartan-Eilenberg resolution $I^{\bullet\bullet}$ of $A^\bullet$ is an upper half plane double complex ($I^{pq}=0$ if $q<0$) consisting of injective objects of $\mathcal{A}$, together with an "augmentation" $A^\bullet \to I^{\bullet\bullet}$ s.t. $\forall p$:

1. $A^p = 0 \Rightarrow I^{p\bullet}$ zero
2. The maps $B^p(\varepsilon): B^p(I, d^h) \leftarrow B^p(A^\bullet)$ are injective resolutions
   $H^p(\varepsilon): H^p(I, d^h) \leftarrow H^p(A^\bullet)$

One can show that $A^\bullet \to \mathrm{Tot}^{\oplus}(I)$ is a quasi-iso.

Lemma (Dual to 5.7.2 in [Weibel]).

Every cochain complex has a C-E resolution.

Proof: $\forall p$ Pick injective resolutions of $B^p(A^\bullet)$ and $H^p(A^\bullet)$, apply Horseshoe to $0 \to B^p \to Z^p \to H^p \to 0$ to get a SES of complexes $0 \to I_B^{p\bullet} \to I_Z^{p\bullet} \to I_H^{p\bullet} \to 0$.

Apply Horseshoe again to $0 \to Z^p \to A^p \to B^{p+1} \to 0$ to get an inj res $0 \to I_Z^{p\bullet} \to I_A^{p\bullet} \to I_B^{p+1\,\bullet} \to 0$.

Now define $I^{\bullet\bullet}$ to be double complex I_A^{pq}, horizontal differential is composite $I^A_{p-1,\bullet} \to I^B_{p\bullet} \hookrightarrow I^Z_{p\bullet} \hookrightarrow I^A_{p\bullet}$. Other props can be checked.

Now we can also define $\mathbb{R}^i F(M^\bullet) = H^i(F(\mathrm{Tot}^{\oplus}(I)))$ for $M^\bullet \to I$ a CE resolution, called right hyper-derived functors.

4. Spectral sequence of a filtered complex

$\cdots F^p A \hookrightarrow F^{p-1} A \hookrightarrow \cdots \hookrightarrow F^0 A = A$

$(A^\bullet, d)$ a complex, a decreasing filtration on $A^\bullet$ is a decreasing filtration $F^p A^k$ on each A^k s.t. $d(F^p A^k) \subseteq F^p A^{k+1}$ $\rightsquigarrow$ subcomplexes $F^p A^\bullet$ of $A^\bullet$. Will assume $\forall k\, \exists l$ s.t. $F^l A^k = 0$.

$\rightsquigarrow$ Filtration $F^p H^i(A^\bullet) = \mathrm{Im}\,(H^i(F^p A^\bullet) \to H^i(A^\bullet))$.

Associated graded $\mathrm{Gr}_p^F H^i(A^\bullet) = \dfrac{F^p H^i(A^\bullet)}{F^{p+1} H^i(A^\bullet)}$

Theorem: There exist complexes $(E_r^{p,q}, d_r)$; $d_r: E_r^{p,q} \to E_r^{p+r, q-r+1}$ satisfying the conditions:

(i) $E_0^{p,q} = \mathrm{Gr}_p^F A^{p+q} := F^p A^{p+q} / F^{p+1} A^{p+q}$ and d_0 induced by d.

(ii) $E_{r+1}^{p,q}$ can be identified with the cohomology of $(E_r^{p,q}, d_r)$ i.e. with

$$\mathrm{Ker}\,(d_r: E_r^{p,q} \to E_r^{p+r, q-r+1}) / \mathrm{Im}\,(d_r: E_r^{p-r, q+r-1} \to E_r^{p,q}).$$

(iii) For p,q fixed and r sufficiently large, we have $E_r^{p,q} = \mathrm{Gr}_p^F H^{p+q}(A^\bullet)$.

5. Two spectral sequences of a double complex & morphisms

For a double complex $C^{\bullet\bullet}$, we can filter $\mathrm{Tot}(C)$ by the columns letting ${}^{I}F^{n}\mathrm{Tot}(C)$ be the total complex of the double subcomplex

$$({}^{I}\tau_{\geq n} C) = \begin{cases} C^{pq} & \text{if } p \geq n \\ 0 & \text{if } p < n \end{cases} \qquad \begin{array}{ccc|cc} \cdots & 0 & 0 & * & * \\ \cdots & 0 & 0 & * & * \\ \cdots & 0 & 0 & * & * \end{array}$$

Then ${}^{I}E_1^{pq} = H^q(C^{p\bullet}, d^v)$, and $d_1 : H^q(C^{p\bullet}, d^v) \to H^q(C^{p+1\,\bullet}, d^v)$ is induced by d^h on C, so we write ${}^{I}E_2^{pq} = H^p_h H^q_v(C)$.

Filtering by rows, we get double complexes $\begin{array}{ccc} * & * & * \\ * & * & * \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ and a spectral sequence with ${}^{II}E_1^{pq} = H^q(C^{\bullet p}, d^h)$

and $${}^{II}E_2^{pq} = H^p_v H^q_h(C).$$

Both of these converge to $H^{p+q}(\mathrm{Tot}(C))$.

Prop [Voisin] Morphism of complexes $\phi : M^\bullet \to K^\bullet$

$\rightsquigarrow$ morphism of CE complexes $\Phi^{\bullet\bullet} : I_M \to I_K$

$\rightsquigarrow$ Morphism of spectral sequences, canonical from E_2 and $\mathbb{R}^k F(M^\bullet) \to R^k F(K^\bullet)$.

compatible with filtrations, converges to the ϕ_∞ morphism

6. Grothendieck spectral sequence

$\mathcal{A}, \mathcal{B}, \mathcal{E}$ abelian categories (enough inj) $\quad G : \mathcal{A} \to \mathcal{B}$, $F : \mathcal{B} \to \mathcal{E}$ left exact

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[dr, "FG"'] & \mathcal{B} \arrow[d, "F"] \\ & \mathcal{E} \end{tikzcd}$$

Defn: $B \in \mathcal{B}$ is F-acyclic if $R^i F(B) = 0$ for $i \neq 0$.

Grothendieck Spectral Sequence

Suppose G sends injectives of $\mathcal{A}$ to F-acyclics of $\mathcal{B}$. Then $\exists$ convergent first quadrant cohom. spec. sequence $\forall A \in \mathcal{A}$.

$${}^{II}E_2^{pq} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof: $A \to I^\bullet$ inj res

$G(I)$ cochain complex $\rightsquigarrow$ CE resolution

2 spectral seqns $\rightsquigarrow$ ${}^{I}E_2^{pq} = H^p((R^qF)(GI)) \Rightarrow (R^{p+q}F)(GI)$

each $G(I^p)$ is F-acyc, so $(R^qF)(G(I^p)) = 0$ for $q \neq 0$,

so collapse to

$$(R^pF)(GI) = H^p(FG(I)) = R^p(FG)(A)$$

$$\rightsquigarrow \quad {}^{II}E_2^{pq} = (R^pF)(\underbrace{H^q(G(I))}_{=\, R^qG(A)}) \Rightarrow R^p(FG)(A)$$

$\square$

7. Leray Spectral sequence

Apply the Grothendieck sp. seq. to $f: X \to Y$ continuous map of top. spaces.

$$\begin{tikzcd} \text{Sheaves}(X) \arrow[r, "f_*"] \arrow[dr, "\Gamma"'] & \text{Sheaves}(Y) \arrow[d, "\Gamma"] \\ & Ab \end{tikzcd}$$

$$(f_*\mathcal{F})(Y) = \mathcal{F}(f^{-1}(Y)) = \mathcal{F}(X).$$

$$R^{p+q}\Gamma = H^{p+q}$$

So

$$E_2^{pq} = (R^p\Gamma)(R^q f_*\mathcal{F}) \Rightarrow R^{p+q}\Gamma(\mathcal{F})$$

$$(R^p\Gamma)(R^q f_*\mathcal{F}) = H^p(Y, R^q f_*\mathcal{F}), \qquad R^{p+q}\Gamma(\mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

Cup product gives morphisms of spectral sequences