K3 Seminar

$X/\mathbb{C}$ smooth & proper variety of dim n $\longrightarrow$ smooth compact complex manifold $X(\mathbb{C})$ using Euclidean topology. of real dim $2n$.

On $X(\mathbb{C})$ we have sheaves $\mathcal{E}_X^{p}$ for the space of complex-valued ~~smooth~~ k-forms

~~Chart~~ $(U, x_1, x_2, \dots, x_{2n})$ $\quad \mathcal{E}_X^p(U) = \langle dx_{i_1} \wedge \dots \wedge dx_{i_p} \mid i_1 < \dots < i_p \rangle_{\mathcal{E}^\infty(U;\mathbb{C})}$.

$$d : \mathcal{E}_X^p(U) \to \mathcal{E}_X^{p+1}(U) \qquad d\Big(\sum_{i_1<\dots<i_k} f_{i_1\dots i_p}\, dx_{i_1} \wedge \dots \wedge dx_{i_p}\Big)$$

$$= \sum_{i_1<\dots<i_k} \sum_j \frac{\partial f_{i_1\dots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p})$$

Get de Rham complex $(\mathcal{E}_X^\bullet(X), d)$, and we define the de Rham cohomology $H^k_{dR}(X)$ to be $H^k(\mathcal{E}^\bullet(X), d)$. Moreover the sequence

$$0 \to \mathbb{C}_X \to \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \mathcal{E}_X^2 \to \cdots \text{ is an acyclic resolution.}$$

De Rham's theorem: $\quad H^k_{dR}(X) = H^k(X; \mathbb{C})$

[Manifolds can be given metrics! Complex manifolds can be given hermitian metrics]

Inside $\mathcal{E}_X^p$, there is the sheaf of holomorphic p-forms Ω_X^p. For $(U, z_1, \dots, z_n)$ holomorphic chart, define $\Omega_X^p(U) = \langle dz_{i_1} \wedge \dots \wedge dz_{i_p} \mid i_1 < \dots < i_p \rangle_{\mathcal{O}_X(U)}$ $\longleftarrow$ holomorphic functions.

Define $\mathcal{E}_X^{(p,0)}$ to be the $\mathcal{C}^\infty_{\mathbb{C}}$ submodule of $\mathcal{E}_X^p$ gen. by Ω_X^p, i.e. $\langle dz_I \rangle_{\mathcal{C}^\infty_{\mathbb{C}}(U)}$ in local coords

and $\mathcal{E}_X^{(0,p)} = \overline{\mathcal{E}_X^{(p,0)}}$ and $\mathcal{E}_X^{(p,q)} = \mathcal{E}_X^{(p,0)} \wedge \mathcal{E}_X^{(0,q)}$. In local coords write $\sum f_{IJ}\, dz_I \wedge d\bar{z}_J$.

Locally define $\partial : \mathcal{E}_X^{(p,q)} \to \mathcal{E}_X^{(p+1,q)}$ $\quad \partial\big(\sum_{I,J} f_{IJ}\, dz_I \wedge d\bar{z}_J\big) = \sum_{I,J} \sum_{i=1}^n \frac{\partial f_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J$

$\bar\partial : \mathcal{E}_X^{(p,q)} \to \mathcal{E}_X^{(p,q+1)}$ $\quad \bar\partial(\qquad) = \text{---} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j \wedge \dots$

and we can see that $d = \partial + \bar\partial$, $\partial^2 = \bar\partial^2 = 0$, $\partial\bar\partial + \bar\partial\partial = 0$ gives Dolbeault cohomology $H^{p,q}_{\bar\partial}(X)$

$\bar\partial$-Poincaré lemma: $D \subset \mathbb{C}^n$ polydisk (i.e. product of disks), then $\forall \alpha \in \mathcal{E}^{(p,q)}(D)$ with $\bar\partial\alpha = 0$, $\exists \beta \in \mathcal{E}_X^{(p,q-1)}(D)$ s.t. $\alpha = \bar\partial\beta$. i.e. $H^{p,q}_{\bar\partial}(D) = 0$ $q \geq 1$.

Dolbeault's theorem: For any complex manifold X, $0 \to \Omega_X^p \to \mathcal{E}_X^{(p,0)} \xrightarrow{\bar\partial} \mathcal{E}_X^{(p,1)} \xrightarrow{\bar\partial} \cdots$ is a soft resolution, and $H^q(X, \Omega_X^p) = H^{p,q}_{\bar\partial}(X)$

Pf: Only need $\Omega_X^p = \ker \bar\partial$ on $\mathcal{E}_X^{(p,0)}$ $\quad \bar\partial\big(\sum_I f_I\, dz_I\big) = \sum_I \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I = 0 \Rightarrow f$ holom.

Have used complex structure, but can do even better using <u>Hermitian metric</u>. Can pick a global section $H \in \Gamma(X, \mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)})$ s.t. in local coords $H = \sum h_{ij}\, dz_i \otimes d\bar{z}_j$ with (h_{ij}) positive definite Hermitian. $(h_{ij} = \overline{h_{ji}})$. And we have a <u>Kähler form</u> $\omega \in \mathcal{E}_X^{(1,1)}$ given in coords by $\omega = \frac{\sqrt{-1}}{2} \sum h_{ij}\, dz_i \wedge d\bar{z}_j$. The <u>volume form</u> $dvol \in \mathcal{E}_X^{(n,n)} = \mathcal{E}_X^{2n}$ is $dvol = \frac{\omega^n}{n!}$.

Then the <u>Hodge star operator</u> ~~is defined by~~ $*: \mathcal{E}_X^p(X) \to \mathcal{E}_X^{2n-p}(X)$ is defined by $\underset{\mathcal{E}^p}{\alpha} \wedge \underset{\mathcal{E}^{2n-p}}{*\beta} = \underset{C^\infty_X}{H(\alpha,\beta)}\, dvol$.
defined locally by $\eta = \sum_{I,J} \eta_{I,J} \varphi_I \wedge \bar{\varphi}_J$ going to $*\eta = 2^{p+q-n} \sum_{I,J} (\text{sign}) \varphi_{I^c} \wedge \bar{\varphi}_{J^c}$.

Use this to define adjoints to the operators $d, \partial, \bar\partial$ given by $d^* = -*d*$, $\partial^* = -*\bar\partial*$, $\bar\partial^* = -*\partial*$. $\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle$ $\to$ <u>Laplacians</u> $\Delta_d = d^*d + dd^*$, $\Delta_\partial = \partial^*\partial + \partial\partial^*$, $\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$

Forms α such that $\Delta_d \alpha = 0$ (resp. $\Delta_{\bar\partial}\alpha = 0$) are called *harmonic* (resp. $\bar\partial$-*harmonic*). They are denoted $\mathcal{H}^i_d(X)$ (resp. $\mathcal{H}^{p,q}_{\bar\partial}(X)$).

Two versions of the Hodge theorem

<u>Hodge theorem</u>: Let X be a compact oriented riemannian manifold. Then every deRham cohomology class has a harmonic representative, i.e. $[\beta] \in H^k_{dR}(X;\mathbb{C})$ then $\exists \alpha \in \mathcal{E}^k_X(X)$ s.t. $\Delta_d \alpha = 0$ and $[\alpha] = [\beta]$.

<u>$\bar\partial$-Hodge theorem</u>: Let X be a compact complex mfld. Then every Dolbeault cohomology class has a $\bar\partial$-harmonic representative.

In general, harmonic and $\bar\partial$-harmonic have nothing to do with each other. However, if we assume that X is <u>Kähler</u>, i.e. $d\omega = 0$, very nice things happen. Get <u>Kähler identities</u>: Let $L = \omega\wedge$ $(1,1)$, $\Lambda = -*L*$ $(-1,-1)$
Then $[\Lambda, \bar\partial] = -\sqrt{-1}\,\partial^*$, $[\Lambda, \partial] = \sqrt{-1}\,\bar\partial^*$. Then one tricks around and shows that $\partial\bar\partial^* + \bar\partial^*\partial = 0$, $\partial^*\bar\partial + \bar\partial\partial^* = 0$. Thus $\Delta = dd^* + d^*d = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial) = \Delta_\partial + \Delta_{\bar\partial}$ loses all cross terms, and one checks that $\Delta_\partial = \Delta_{\bar\partial}$. Thus,

<u>Theorem</u> For X a Kähler manifold, $\Delta_d = 2\Delta_{\bar\partial}$.

Then we obtain the Hodge decomposition:

<u>The Hodge decomposition</u>: Suppose X compact Kähler. Then a diff. form is harmonic iff its (p,q) components are. Consequently, we have isomorphisms

$$H^i(X;\mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega^p_X).$$

Furthermore, complex conjugation induces $\mathbb{R}$-linear isomorphisms between the harmonic (p,q)- and (q,p)-forms. Therefore $H^q(X,\Omega^p_X) \cong H^p(X, \Omega^q_X)$.

<u>Proof</u>: Write $\mathcal{E}^i_X(X) = \bigoplus_{p+q=i} \mathcal{E}^{(p,q)}_X(X)$. Since $\Delta_{\bar\partial}$ does not interchange the parts of this decomposition, thus neither does Δ. Thus $H^i(X;\mathbb{C}) \cong \bigoplus_{p+q=i} \mathcal{H}^{p,q}_d(X)$. Moreover, since $\Delta_d = 2\Delta_{\bar\partial}$, $\mathcal{H}^{p,q}_d(X) \cong \mathcal{H}^{p,q}_{\bar\partial}(X)$.
Thus $H^i(X;\mathbb{C}) \cong \bigoplus_{p+q=i} \mathcal{H}^{p,q}_{\bar\partial}(X) \cong \bigoplus_{p+q=i} H^q(X,\Omega^p_X)$ by Dolbeault's theorem.
Complex conjugation induces an iso $\mathcal{H}^{p,q}_d \cong \mathcal{H}^{q,p}_d$.