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Fundamental groups in real and complex algebraic geometry

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The problem with the word problem

Finitely presented groups

A group G is finitely presentable if there is a finite set of generators $S = \{x_1, \ldots, x_n\}$ and a finite set of relations $R = \{r_1, \ldots, r_k\}$, such that

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Non-example:

- Any uncountable group, e.g. $(\mathbb{R}, +)$.

The word problem

Problem (The word problem for finitely presented groups)

Given $G = \langle S | R \rangle$ *and an unreduced word* $w = x_{w_1}^{e_1} \dots x_{w_j}^{e_k}$, $e_i = \pm 1$, *is there an algorithm that can decide if* $w = 1$ *after using the relations?*

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Example solution for $G = \langle x, y | \emptyset \rangle$ and some word w.

1. Eliminate all pairs xx^{-1}, $x^{-1}x$, yy^{-1}, $\dots$, in w.
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Theorem (Novikov-Boone-Britton)

There exists a finitely presented group G with an unsolvable word problem.

No algorithm to determine whether two finitely presented groups are isomorphic! [1, Prop 12.34].

Fundamental groups of manifolds

Smooth manifolds

Theorem

For all finitely presented groups $G = \langle S|R \rangle$, there is a smooth closed n-manifold M with $\pi_1(M) = G$, for every $n \geq 4$.

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Let G have $|S| = k$, $|R| = l$. Take

$$X = (S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1}),$$

k-times.
The relations can be represented by l-smoothly embedded circles.
By surgery, replace the neighbourhoods $S^1 \times D^{n-1}$ by $D^2 \times S^{n-2}$, giving a smooth closed manifold M.
Then the Seifert-van Kampen theorem implies $\pi_1(M) \cong G$.
This has been known at least since the 1930s.

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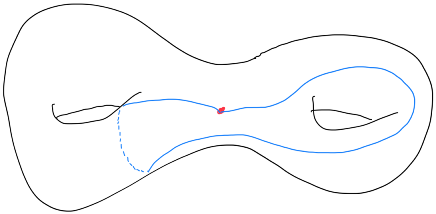

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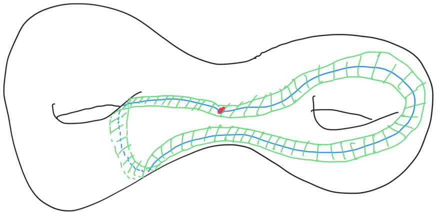

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For all finitely presented groups $G = \langle S|R \rangle$, there is a smooth closed n-manifold M with $\pi_1(M) = G$, for every $n \geq 4$.

Corollary

It is impossible to classify manifolds up to diffeomorphism.

Such a classification would solve the word problem!

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- Solve a particular PDE on vector bundles of X to get a complex 3-manifold Y.

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Some finitely presented group G cannot be the fundamental group of a closed Kähler manifold.

This is in part because the existence of the Kähler metric forces a decomposition:

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But, for any finitely presented group G, there is an open Kähler surface with $\pi_1(X) = G$. For details see [2, Chapter 1.].

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Some non-examples:

- Non-trivial free groups.
- Groups where the rank of the Abelianisation is not even.
- $SL(N, \mathbb{Z})$ is not Kähler, for all $N \geq 2$.

Fundamental groups of varieties

Real varieties

Let $X = V(f_1, \dots, f_r) \subseteq \mathbb{P}^n_{\mathbb{C}}$ be the zero set of $f_i \in \mathbb{R}[x_0, \dots, x_n]$. The real points is the set

$$X(\mathbb{R}) = \{x \in \mathbb{RP}^n | \ f_i(x) = 0, \ \forall i\}$$

with the Euclidean topology.

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For every compact differentiable manifold M, there is a smooth, projective real variety X such that M is diffeomorphic to $X(\mathbb{R})$.

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Corollary

No algorithm can determine whether two smooth projective real varieties are isomorphic.

Complex varieties

Let $X = V(f_1, \dots, f_r) \subseteq \mathbb{P}^n_{\mathbb{C}}$ be the zero set of $f_i \in \mathbb{C}[x_0, \dots, x_n]$. The complex points is the set

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What is the possible topology of $X(\mathbb{C})$?

NB: In general a hard problem, see for instance the Hodge conjecture.

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Let G be a finitely presented group. Then there is an irreducible complex singular projective variety X such that $\pi_1(X(\mathbb{C})) = G$.

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Theorem (Simpson [7])

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Again, no algorithm can exist to classify singular complex projective varieties.

Smooth complex varieties

What if $X = V(f_1, \ldots, f_r) \subseteq \mathbb{P}^n_{\mathbb{C}}$ is smooth? Then $X(\mathbb{C})$ is a Kähler manifold, hence $\pi_1(X(\mathbb{C}))$ is restricted to be a Kähler group.

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Theorem (Lefschetz Hyperplane Theorem)

Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective variety, and $Y = X \cap H$ a generic hyperplane section. Then for all $i \leq \dim_{\mathbb{C}}(X) - 2$, we have

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Open problem

Is every Kähler group the fundamental group of a smooth projective complex variety?

$\exists$ compact Kähler manifolds which do not have the homotopy type of such varieties [8].

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